A New Nested MIMO Array With Increased Degrees of Freedom and Hole-Free Difference Coarray

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***Abstract*—We propose a new antenna array design approach for a multiple-input and multiple-output (MIMO) radar, which has closed-form expressions for the sensor locations and the number of achievable degrees of freedom (DOFs). This new approach utilizes the nested array as transmitting and receiving arrays. We employ the difference coarray of the sum coarray (DCSC) of the MIMO radar to obtain more DOFs for direction-of-arrival (DOA) estimation. Via properly designing the interelement spacings of the transmitting and receiving arrays, we can obtain a hole-free DCSC. The characteristics of array geometries are analyzed and the optimal numbers of sensors in transmitting/receiving antenna array are derived when given the total number of physical sensors. Simulations are conducted to demonstrate the advantages of the proposed array in terms of the number of DOFs, the number of resolvable sources, and the DOA estimation performance over the coprime MIMO array.**

***Index Terms*—Difference coarray, minimum redundancy array (MRA), multiple-input and multiple-output (MIMO) radar, nested array, sensor arrays, sum coarray.**

I. Introduction

MULTIPLE-INPUT and multiple-output (MIMO) radars have been a subject of intense research in recent years. By transmitting orthogonal or noncoherent waveforms and extracting the waveforms at each receiving element with a set of matched filters, a virtual array with larger aperture and more degrees of freedom (DOFs) can be synthesized [1], [2]. For colocated transmitting and receiving antennas, the MIMO radar has been shown to provide higher resolution and better parameter estimation/identification performance.

Sparse array design with closed-form expressions of sensor locations has been a hot topic, and some new arrays, such as the nested array [3], [4], the coprime array [5], [6], and the nested minimum redundancy array (MRA) [7], have been proposed recently. In order to fully exploit the freedom provided by the MIMO radar, Qin *et al.* utilize a coprime pair of uniform linear arrays (ULA) within the MIMO radar framework [8], in which the array is examined in a sum coarray context and can achieve a better direction-of-arrival (DOA) estimation performance. In a similar way, another nested MIMO array structure was proposed in [9] by using a dense ULA as the transmitting array and a sparse ULA as the receiving array. Since only the sum coarray was employed in the two aforementioned MIMO arrays, both of them can provide a limited number of DOFs. To further increase the DOFs in an MIMO radar, other approaches [10], [11] take the difference coarray of the sum coarray (DCSC) into consideration and obtain a higher number of DOFs. Two different nonuniform array geometries have been studied. The first one is the minimum redundancy (MR) MIMO array [11], [12], which can provide as many as possible DOFs in the perspective of the DCSC. Unfortunately, optimization of this kind of array geometry usually involves large computational loads [12]–[15]. The second geometry is the transmitting/receiving coprime array MIMO radar [10], which uses a part or the entire coprime array for transmission and reception. It is desired that the difference coarray is a hole-free ULA, and thus, the spatial smoothing (SS) algorithm can be employed to decorrelate the signal for DOA estimation [3], while the DCSC is not a hole-free ULA for the second array geometry.

In order to overcome these limitations, we propose a new nested MIMO array design approach in this letter utilizing the nested array [3] as the transmitting/receiving array. First, we use an entire nested array as the transmitting array in an MIMO radar. Then, we calculate the consecutive number of elements in the difference coarray of the transmitting array. Finally, the receiving array can be constructed by another nested array with a larger unit interelement spacing D.

The specific contributions of this letter are as follows.

1) By properly designing D, we can obtain a difference coarray with relatively larger aperture and more DOFs from the sum coarray of the MIMO radar. Meanwhile, the DCSC is a hole-free ULA. Although the transmitting/receiving coprime MIMO radar [10] also utilizes the DCSC to increase DOF and to estimate DOA, its achievable DOF and DOA estimation performance is much lower than our proposed array, as demonstrated in Sections III-C and IV. In addition, there are holes in the DCSC of the coprime MIMO radar.
2) Thanks to the closed-form expressions for sensor locations in the nested array, our proposed design approach for the MIMO radar also enjoys closed-form expressions for both sensor locations and the number of DOFs. By contrast,

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the MR MIMO radar designed in [11]–[15] fails to give closed-form expressions and needs a significant amount of computational complexity.

3) We analyze the characteristics of the array geometries and derive the optimal number of sensors in the transmitting and receiving arrays provided the total number of physical sensors. As a result, we can design a large MIMO radar array easily without any search or combinatorial methods.

II. Signal Model

We consider a colocated MIMO radar system with M transmitters and N receivers. The transmitting array and the receiving array are both nested arrays with their locations denoted by $\{u_{T,m}\}_{m=1}^{M}$ and $\{u_{R,n}\}_{n=1}^{N}$, respectively, where $\{u_{T,m}\}$ and $\{u_{R,n}\}$ are integers normalized by a half wavelength of the incident signal. By transmitting orthogonal waves from each transmitter and extracting the waveforms at each receiver using a series of matched filters, a virtual array of MN elements can be synthesized with locations [1], [2]

$$\{u_k\} = \{u_{T,m} + u_{R,n} | m = 1, 2, \ldots, M; n = 1, 2, \ldots, N\} \quad (1)$$

where $k = 1, 2, \ldots, MN$. From (1), we can see that the locations of the virtual array of an MIMO radar are the cross summation of the locations of the transmitting array and the receiving array. Therefore, we refer to this virtual array as the "sum coarray" of the MIMO radar.

In order to obtain the DCSC, we would check the spacing set of $\{u_k - u_{k'}\}$, which can be expressed as

$$\begin{aligned}\{u_{T,m} + u_{R,n} - u_{T,m'} - u_{R,n'} | m, m' = 1, 2, \ldots, M;\\ n, n' = 1, 2, \ldots, N\}.\end{aligned} \quad (2)$$

These spacings $\{u_k - u_{k'}\}$ are corresponding to the element locations of a difference coarray [3], [16] obtained by vectorizing the covariance matrix of the equivalent received data from the sum coarray of an MIMO radar. To further increase the DOF and obtain a hole-free difference coarray, the locations of the transmitting and receiving arrays of an MIMO radar can be implemented by solving the following optimization problem [12]:

$$\begin{aligned}&\max_{\{u_{T,m}\},\{u_{R,n}\}} \; L\\ &\text{s.t. } |u_{T,m}| = M, \; |u_{R,n}| = N,\\ &\quad \{u_{T,m} + u_{R,n} - u_{T,m'} - u_{R,n'}\}\\ &\quad \supset \{-L, \ldots, -1, 0, 1, \ldots, L\},\end{aligned} \quad (3)$$

where L is the largest contiguous aperture in the difference coarray of the MIMO radar, and $|u|$ denotes the cardinality of the set u.

Solving this optimization problem usually requires an exhaustive search algorithm [12]. Various relax algorithms based on difference bases (DS), cyclic DS [16], statistical optimization [14], [17], and MRA [18] have been proposed to obtain relatively easy design approaches. However, the number of available DSs is relatively small, the statistical optimization algorithms suffer from large computational costs, and the number of known MRAs is limited. To address these challenges, in the following, we propose a new nested MIMO design approach, which utilizes the existing nested arrays and enjoys closed-form expressions for the element locations and the number of DOF.

III. Proposed Nested MIMO Array

A. Array Geometry

The proposed nested MIMO array is constructed by the following two steps.

1) We use an M-element nested array as the transmitting array, whose locations are denoted by

$$\{u_{T,m}\} = \{a_m | m = 1, 2, \ldots, M\} \quad (4)$$

where $\{a_m\}_{m=1}^{M}$ is the location of the nested array. In order to obtain a hole-free difference coarray, we use a two-level nested array (refer to as the nested array hereafter) in this letter. The number of DOFs in the difference coarray of the nested array is f_M, which can be obtained using relevant equations from [3]

$$f_M = \begin{cases} M^2/2 + M - 1, & \text{if } M \text{ is even} \\ (M+1)^2/2 - 1, & \text{if } M \text{ is odd.} \end{cases} \quad (5)$$

2) The receiving array can be constructed by another nested array with a larger unit interelement spacing D, whose locations can be denoted by

$$\{u_{R,n}\} = \{b_n \cdot D | n = 1, 2, \ldots, N\} \quad (6)$$

where $\{b_n\}_{n=1}^{N}$ is the location sequence of the nested array. Substituting (4) and (6) into (1), we can obtain the location set of the sum coarray of the MIMO radar

$$\begin{aligned}\{u_{T,m} + u_{R,n}\} = \{a_m + b_n \cdot D \; | m = 1, 2, \ldots, M,\\ n = 1, 2, \ldots, N\}.\end{aligned} \quad (7)$$

The location set of the DCSC can be expressed as

$$\begin{aligned}&\{a_m + b_n \cdot D - a_{m'} - b_{n'} \cdot D\} = \{(a_m - a_{m'})\\ &\quad + (b_n - b_{n'}) \cdot D \; |\\ &m, m' = 1, 2, \ldots, M; n, n' = 1, 2, \ldots, N\}.\end{aligned} \quad (8)$$

The two terms $(a_m - a_{m'})$ and $(b_n - b_{n'})$ are the location sets of the difference coarray of the transmitting array and the receiving array, respectively. In this way, as long as D is equal to f_M, it is easy to prove that we will obtain a filled ULA in the DCSC of the MIMO radar [7]. Thanks to the fact that the transmitting and receiving arrays have closed-form expressions, the sum coarray and the DCSC of the proposed nested MIMO radar array also embed closed-form expressions.

An example of the proposed nested MIMO radar with a four-element transmitting array and a five-element receiving array is illustrated in Fig. 1, where only the nonnegative part of the DCSC is plotted due to the symmetric property of the positions of the difference coarray [3]. Owing to the fact that the aperture length of the difference coarray of the transmitting array is 11, the unit interelement spacing $D = 11$ in the receiving array. From Fig. 1, we know that the aperture length of the sum coarray $L = 93$ in this example, and the DCSC is a hole-free ULA.

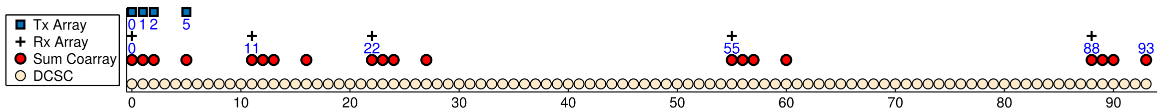

Fig. 1. Example of the nested MIMO radar array with a four-element transmitting (Tx) array and a five-element receiving (Rx) array, in which $D = 11$. Only the nonnegative part of the DCSC is given.

B. Properties of the Nested MIMO Array

Proposition 1: The number of DOFs of the proposed nested MIMO array in the DCSC is

$$f = f_M \cdot f_N \tag{9}$$

where f_M and f_N are the number of DOFs of the transmitting and receiving arrays, respectively.

Proof. The difference coarray of a nested array with M (or N) sensors is a filled ULA [3], and hence, the number of DOFs f_M (or f_N) and its aperture length l_M (or l_N) satisfy

$$f_M = 2l_M + 1 \text{ (} or \text{ } f_N = 2l_N + 1 \text{).} \tag{10}$$

In our nested MIMO array design, the interelement spacing D in the receiving array is set to $D = f_M$. Thus, we can obtain the aperture length of the sum coarray using (7)

$$L_{\text{SCA}} = l_M + l_N \cdot D = l_M + l_N \cdot f_M \tag{11}$$

where l_N is the aperture length of the receiving nested array with N elements excluding interelement D.

Because the DCSC of the MIMO array with this geometry is also a filled ULA, the number of DOFs in the DCSC is

$$\begin{aligned} f &= 2L_{SCA} + 1 \quad = 2l_M + 2l_N \cdot f_M + 1 \\ &= f_M + 2l_N \cdot f_M = f_M \cdot (2l_N + 1) \\ &= f_M \cdot f_N. \end{aligned} \tag{12}$$

■

Following this, we can attain $f_M \cdot f_N$ freedoms in the DCSC using only $M + N$ elements for the nested MIMO array. There are closed-form expressions for the element locations of the transmitting array, the receiving array, the sum coarray and the DCSC, as well as the number of freedoms in the DCSC. It is worth noting that these are not using the current available constructing approaches for MR MIMO radar based on the DS, cyclic DS [16], and MRA [18]. Under the constraint of fixed total number of physical sensors, i.e., $M + N = K$, we can further optimize the distribution of transmitting and receiving arrays by finding M and N that maximize the total number of DOFs. By using the arithmetic mean–geometric mean inequalities, the optimal value of M and N can be obtained by

$$\begin{cases} M = N = K/2, & \text{if } K \text{ is even} \\ M = (K-1)/2, N = (K+1)/2, & \text{if } K \text{ is odd.} \end{cases} \tag{13}$$

Once M and N are determined, we can obtain the optimal geometry of the transmitting/receiving nested array using the relevant conclusion in [3]. Let us the take the following case as an example. When K, M, and N are all even numbers, $M = N = K/2$ according to (13), we can calculate the optimal sensor numbers of N_1 and N_2 in the nested array, which are equal to $N/2 = K/4$. Therefore, the number of DOFs can be obtained by (9)

$$\begin{aligned} f &= f_M \cdot f_N \\ &= (M^2/2 + M - 1) \cdot (N^2/2 + N - 1) \\ &= (K^2/8 + K/2 - 1)^2. \end{aligned} \tag{14}$$

Thereby, it can be seen that our proposed nested MIMO array can provide $O(K^4)$ DOFs using only K sensors. The number of DOFs for other cases when K, M, and N are odd or even number combinations can be derived in a similar way using (13) and relevant equations in [3].

Therefore, given a total number of physical sensors K, we can obtain an optimal nested MIMO array instantly almost requiring no complex calculation. The computational cost will not vary with the increase of physical sensors. Hence, it is much easier to construct a larger MIMO radar using this approach.

C. Relationship and Difference With the Existing MIMO Arrays

The coprime MIMO radar proposed in [8] uses a coprime pair ULAs as the transmitting array and the receiving array, and the nested MIMO array in [9] uses a dense ULA as the transmitting array and a sparse array as the receiving array. In addition, only the sum coarrays of the MIMO radar are employed for DOA estimation for the former two arrays. By contrast, our proposed nested MIMO array uses an entire nested array as the transmitting array and another nested array with larger interelement spacing as the receiving array. Furthermore, we use the DCSC of the MIMO radar for DOA estimation, which can provide more DOFs and larger aperture. Therefore, the new nested MIMO array is different from our proposed nested MIMO array proposed in [9].

According to (3), the MR MIMO array is designed to obtain a contiguous aperture as large as possible in the DCSC of an MIMO radar. Therefore, the MR MIMO arrays constructed by DS, cyclic DS, almost DS [16], [19], and MRA [18] have more DOFs than our proposed nested MIMO array. However, our proposed nested MIMO array benefits from closed-form expressions for the element locations and the number of DOFs, while those MR MIMO arrays mentioned above fail to give closed-form expressions.

The transmitting/receiving coprime MIMO radar in [10] uses a part or the entire coprime array for transmission and reception. These geometries can increase the number of DOFs, but the number is not large enough. The number of DOFs can be improved by increasing the interelement spacing D in the receiving array. However, the DCSC of the MIMO radar is not a hole-free ULA.

TABLE I
DOF COMPARISON WITH THE TOTAL NUMBER OF SENSORS (K)

K	6	9	14	18	22	26	30
M_c for coprime MIMO[1]	2	3	4	5	6	7	8
N_c for coprime MIMO	3	4	7	9	11	13	15
DOF of coprime MIMO	35	75	179	293	435	605	803
DOF of nested MIMO	49	187	961	2401	5041	9409	16129

[1] $2M_c + N_c - 1 = K$.

The coprime MIMO array proposed in [10] uses the DCSC for DOA estimation and has a closed-form expression for the sensor locations, which are the same properties as our proposed array. Therefore, we choose it for comparison in the following DOF and DOA estimation comparison.

D. DOF Comparison

We compare the achievable DOFs of our nested MIMO array with the coprime MIMO array in [10]. The coprime array is comprised of a pair of ULAs with $M_c\lambda/2$ and $N_c\lambda/2$ interelement spacing, respectively, where M_c and N_c are coprime integers and λ is the signal wavelength. The coprime MIMO array [10] uses the entire coprime array to transmit and to receive, in which this configuration has the largest DOF in [10]. Let the total number of physical sensors (K) be some integers from 6 to 30, and we choose the optimal number of sensors M_c and N_c in the coprime MIMO array, as listed in Table I. The achievable DOF of the coprime MIMO array is $(7M_c - 3)N_c + M_c$ [10], and the DOF achieved from the nested MIMO array can be calculated by (5), (9), and (13).

We obtain the comparison results for the two array geometries with respect to K, as listed in Table I. It is clear that the number of achievable DOFs increases with K, and our proposed nested MIMO radar array has a significant improvement in DOFs compared with the coprime MIMO array.

IV. NUMERICAL EXAMPLES

In this section, we conduct simulation experiments to show the characteristics of the nested MIMO array and compare with the coprime MIMO array in [10].

First, we show the ability of our proposed nested MIMO array to resolve more sources than sensors using the SS-based MUSIC algorithm [3]. We consider nine physical sensors as an example, whose sum coarray and the DCSC are illustrated in Fig. 1. We also use the coprime MIMO array with nine physical sensors for comparison, whose relevant parameters can be found in Table I. We consider 40 uncorrelated sources impinging on the arrays with equal power, whose spatial frequencies $\sin\theta$ are uniformly positioned range from -0.95 to 0.95. The normalized spatial MUSIC spectra is shown in Fig. 2, where 1000 noise-free snapshots are used. It can be observed that our proposed array geometry can resolve all the 40 sources correctly, but the coprime MIMO array fails.

In order to evaluate the DOA estimation performance of our proposed nested MIMO array, we use Monte Carlo simulations to compute the average root-mean-square error (RMSE) of the estimated DOAs. The RMSEs are averaged over 500 independent Monte Carlo trials. We consider the same two array geometries described in Section IV and use $Q = 5$ narrow-band uncorrelated sources from directions $[-30^\circ, -20^\circ, 30^\circ, 40^\circ, 45^\circ]$. Fig. 3 plots the RMSEs of the two array configurations as a function of SNR, where 500 snapshots are used. It can be seen that our proposed nested MIMO radar geometry achieves better performance than the coprime MIMO array, which demonstrates the superiority of our proposed array design approach.

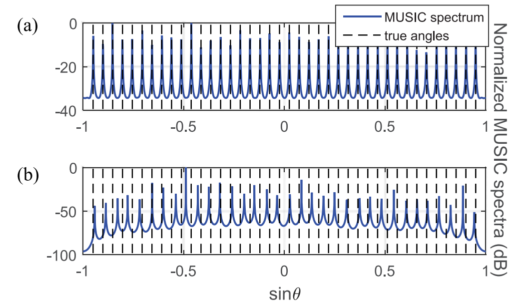

Fig. 2. MUSIC spectra as function of sine of DOA. Vertical dash lines are true positions of sources. (a) Spatial MUSIC spectrum of proposed nested MIMO array. (b) Spatial MUSIC spectrum of the coprime MIMO array.

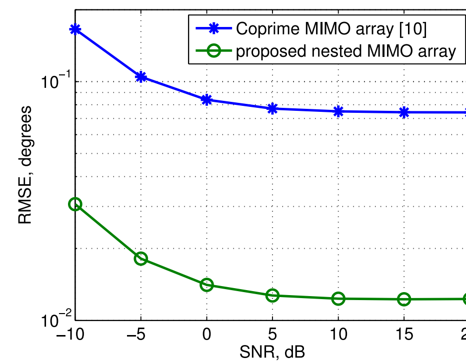

Fig. 3. DOA estimation performances versus SNR for two arrays with nine physical sensors.

V. CONCLUSION

We have proposed a new nested MIMO array design approach utilizing the nested arrays, which features on having a closed-form expression for the sensor locations and the number of achievable DOFs. Furthermore, the DCSC is a hole-free ULA. We have analyzed the characteristics of the nested MIMO array and derived the optimal value of the transmitting/receiving array given a total number of physical sensors. We have also showed the advantages of the new array in terms of the number of DOFs, the number of resolvable sources, and the DOA estimation performance over the coprime MIMO array through simulations. It should be noted that the array design approach in this letter is applicable for all existing sparse arrays with hole-free difference coarray, such as the MRA [20], the super nested arrays [21], the improved nested arrays [22], and so on. Using these arrays to replace the nested array in the design, we will obtain some new MIMO array with similar properties as the nested MIMO array.

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