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Article *in* Journal of Computational Physics · July 2016

DOI: 10.1016/j.jcp.2016.04.039

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Journal of Computational Physics 316 (2016) 614–631

Contents lists available at ScienceDirect

Journal of Computational Physics

www.elsevier.com/locate/jcp

Finite difference methods with non-uniform meshes for nonlinear fractional differential equations ☆

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A R T I C L E I N F O

Article history:
Received 3 October 2015
Received in revised form 17 April 2016
Accepted 18 April 2016
Available online 20 April 2016

Keywords:
Nonlinear fractional differential equation
Non-uniform mesh
Rectangle formula
Trapezoid formula
Predictor–corrector method

A B S T R A C T

In this article, finite difference methods with non-uniform meshes for solving nonlinear fractional differential equations are presented, where the non-equidistant stepsize is non-decreasing. The rectangle formula and trapezoid formula are proposed based on the non-uniform meshes. Combining the above two methods, we then establish the predictor–corrector scheme. The error and stability analysis are carefully investigated. At last, numerical examples are carried out to verify the theoretical analysis. Besides, the comparisons between non-uniform and uniform meshes are given, where the non-uniform meshes show the better performance when dealing with the less smooth problems.

1. Introduction

In recent years, growing attention has been focused on fractional differential equations because they can provide a better approach to describe the complex phenomena in nature, such as viscoelastic materials, anomalous diffusion, signal processing and control theory, etc., see [1,16,17,28,29,32,38]. Compared to classical integer-order differential equations, the theoretical investigations and establishment of numerical schemes for fractional-order (or fractional for brevity) versions are more complicated due to the special properties of fractional differential operators, such as the non-locality, history dependence, and/or long-range interactions [37].

In this paper, we study the numerical schemes for the following nonlinear fractional initial value problem

$$\begin{cases} {}_C D_{0,t}^{\alpha} y(t) = f(t,y), & T \geq t > 0, \\ y^{(k)}(0) = y_0^{(k)}, & k = 0, 1, \cdots, n-1, \end{cases} \tag{1.1}$$

where the continuous function $f(t,y)$ is nonlinear with respect to the unknown function y, n is a positive integer such that $n-1 < \alpha < n$ and the initial values $y_0^{(k)}$ are assumed to be given. Here ${}_C D_{0,t}^{\alpha} y(t)$ is the Caputo derivative, defined by

$${}_C D_{0,t}^{\alpha} y(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} y^{(n)}(s) ds, \quad n-1 < \alpha < n \in Z^+.$$

☆ The work was partially supported by the National Natural Science Foundation of China under Grant no. 11372170, Key Program of Shanghai Municipal Education Commission under Grant no. 12ZZ084 and the grant of "The First-class Discipline of Universities in Shanghai".

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http://dx.doi.org/10.1016/j.jcp.2016.04.039
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In order to ensure the existence of a unique solution of (1.1), we always assume that f satisfies Lipschitz condition with respect to the second variable, that is, $|f(t, y_1) - f(t, y_2)| \le L|y_1 - y_2|$ where $L > 0$. It is well known that the initial value problem (1.1) is equivalent to the following Volterra integral equation [6,16]

$$y(t) = \sum_{j=0}^{n-1} \frac{t^j}{j!} y_0^{(j)} + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, y(s)) ds, \tag{1.2}$$

in the sense that a continuous function is a solution of (1.1) if and only if it solves equation (1.2).

There have existed some studies on the numerical approaches for fractional differential equation (1.1). Early in 1986, Lubich studied the numerical approximation of fractional integrals based on a discrete convolution form and introduced the fractional linear multistep method [26,27]. Almost at the same time, Brunner and Houwen also presented linear multistep methods for Volterra equations [2]. In 1997, Diethelm proposed an implicit algorithm for the fractional differential equations based on a quadrature formula approach in a finite-part integral sense [5]. For more details about the relationship between fractional derivative and finite-part integral, see [12,13]. In 2000, Podlubny proposed a unification discrete form based on triangular strip matrices [33], which was further developed in [34]. In 2002, Diethelm et al. discussed an Adams-type predictor–corrector method for equation (1.2) and gave a detailed error analysis that the convergent order was $\min\{2, 1+\alpha\}$ if ${}_C D_{0,t}^{\alpha} y(t) \in C^2[0, T]$ [7,8]. Shortly after, Li and Tao further studied the error analysis for the fractional predictor–corrector method [20]. And in [22], Li and Zeng reviewed the finite difference methods for fractional ordinary/partial differential equations. In [24], various kinds of numerical methods for fractional differential equations have been studied thoroughly. There have also been some investigations on the stability analysis for nonlinear fractional differential equations. Ladaci and Moulayin [18] analyzed the L^p-stability of fractional nonlinear differential equations. Deng [4] considered the sufficient conditions for the local asymptotical stability of nonlinear fractional differential equations. After that, Li and Zeng [23] studied the numerical stability of the finite difference methods for nonlinear fractional ordinary differential equation (1.1). In very recent, a series of numerical methods for Caputo derivatives is derived and the established algorithms are applied to Caputo-type advection–diffusion equations [3,19,21]. In the meantime, a series of numerical methods for Riesz derivatives is established and applied to Riesz-type fractional partial differential equations [9–11]. Keshtkar et al. [15] investigated the stability of equilibria in the nonlinear fractional-order dynamical systems.

In general, the existence of a weakly singular kernel $(t-s)^{\alpha-1}$ $(0 < \alpha < 1)$ in fractional derivative and integral makes it more difficult to get a higher-order scheme. Particularly when the solution of equation (1.1) is not suitably smooth, those methods on uniform meshes seem to have a poor convergent rate. For these reasons, numerical methods on non-uniform meshes have been placed on the agenda. Especially in recent years, finite difference schemes with non-uniform meshes for fractional differential equations have attracted increasing attention. In [36,40], Yuste and Quintana-Murillo proposed an L1 method with non-uniform timesteps for fractional diffusion and diffusion–wave equations. In [35], Podlubny et al. studied the matrix approach on non-equidistant grids. Mustapha et al. [30,31] used the finite difference method to a sub-diffusion equation. Lopez-Fernandez and Sauter [25] presented a generalized convolution quadrature with variable timesteps. In very recent, Zhang et al. [42] investigated the finite difference scheme for the fractional diffusion equation on non-uniform meshes. Finite difference methods with non-uniform meshes often show great advantages when dealing with less smooth problems. However, the theoretical analysis of stability and convergence of the schemes with non-uniform meshes for nonlinear fractional differential equation seems not to be derived thoroughly yet.

In this paper, we mainly focus on the stability and convergence analysis of three kinds of numerical approaches on non-uniform meshes and illustrate their suitability for non-smooth problems through numerical tests. The rest of the paper is organized as follows. In Section 2, we outline the numerical schemes on the non-uniform meshes. Detailed stability and error analysis for the derived schemes are given in Sections 3 and 4, respectively. In Section 5, numerical examples are carried out to verify the theoretical analysis and to check the capability of the derived methods for non-smooth problems. The conclusions are included in the last section.

2. Numerical schemes on non-uniform meshes

For an integer N and the given time T, we divide the interval $[0, T]$ into $0 = t_0 < t_1 < \cdots < t_k < t_{k+1} < \cdots < t_N = T$, with non-equidistant stepsizes $\tau_i = t_{i+1} - t_i, 0 \le i \le N-1$, and denote $\tau_{max} = \max\limits_{0 \le i \le N-1} \tau_i$, $\tau_{min} = \min\limits_{0 \le i \le N-1} \tau_i$. If the given question is singular at the origin, then the choice of the non-equidistant stepsizes obeys non-decreasing rule, i.e, $\tau_i \le \tau_{i+1}$. Throughout this paper, we do not re-state this if no confusion appears.

Let y_j be the approximate solution of $y(t_j)$ $(j = 0, 1, \cdots, k)$ which have been determined. Now we need to calculate y_{k+1}. We introduce three approaches to do this.

Consider the following integral

$$\begin{aligned}I_{k+1} &= \int_0^{t_{k+1}} (t_{k+1}-s)^{\alpha-1} f(s, y(s))ds \\ &= \sum_{j=0}^{k} \int_{t_j}^{t_{j+1}} (t_{k+1}-s)^{\alpha-1} f(s, y(s))ds,\end{aligned}$$

where $k = 0, 1, \cdots, N-1$.

It can be approximated by the following approach

$$I_{k+1} \approx \sum_{j=0}^{k} \int_{t_j}^{t_{j+1}} (t_{k+1}-s)^{\alpha-1} \tilde{f}_j(s, y(s))ds,$$

where $\tilde{f}_j(s, y(s))$, $j = 0, 1, \cdots, k$ is the approximation of $f(s, y(s))$ on the interval $[t_j, t_{j+1})$.

It will lead to different schemes by choosing different $\tilde{f}_j(s, y(s))$. Here we choose three kinds of $\tilde{f}_j(s, y(s))$ to derive the fractional rectangle, trapezoid, and predictor–corrector methods respectively.

(i) By choosing $\tilde{f}_j(s, y(s))$ as

$$\tilde{f}_j(s, y(s)) = f(t_j, y_j), \quad j = 0, 1, \cdots, k,$$

the fractional rectangle method is derived as

$$y_{k+1} = \sum_{j=0}^{n-1} \frac{t_{k+1}^j}{j!} y_0^{(j)} + \sum_{j=0}^{k} w_{j,k+1} f(t_j, y_j), \tag{2.1}$$

where

$$\begin{aligned}w_{j,k+1} &= \frac{1}{\Gamma(\alpha)} \int_{t_j}^{t_{j+1}} (t_{k+1}-s)^{\alpha-1} ds \\ &= \frac{(t_{k+1}-t_j)^\alpha - (t_{k+1}-t_{j+1})^\alpha}{\Gamma(\alpha+1)}, \quad j = 0, 1, \cdots, k.\end{aligned} \tag{2.2}$$

(ii) If $\tilde{f}_j(s, y(s))$ is selected as

$$\tilde{f}_j(s, y(s)) = \frac{s-t_{j+1}}{t_j-t_{j+1}} f(t_j, y_j) + \frac{s-t_j}{t_{j+1}-t_j} f(t_{j+1}, y_{j+1}), \quad j = 0, 1, \cdots, k,$$

then the fractional trapezoid method is given by

$$y_{k+1} = \sum_{j=0}^{n-1} \frac{t_{k+1}^j}{j!} y_0^{(j)} + \sum_{j=0}^{k+1} \widetilde{w}_{j,k+1} f(t_j, y_j), \tag{2.3}$$

in which

$$\widetilde{w}_{j,k+1} = \frac{1}{\Gamma(\alpha+2)} \begin{cases} \dfrac{1}{t_1} A_0, & \text{if } j = 0, \\ \dfrac{1}{t_{j+1}-t_j} A_j + \dfrac{1}{t_{j-1}-t_j} B_j, & \text{if } j = 1, 2, \cdots, k, \\ (t_{k+1}-t_k)^\alpha, & \text{if } j = k+1, \end{cases} \tag{2.4}$$

and

$$\begin{cases} A_0 = (t_{k+1}-t_1)^{\alpha+1} - t_{k+1}^{\alpha+1} + (\alpha+1) t_1 t_{k+1}^\alpha, \\ A_j = (t_{k+1}-t_{j+1})^{\alpha+1} - (t_{k+1}-t_j)^{\alpha+1} + (\alpha+1)(t_{j+1}-t_j)(t_{k+1}-t_j)^\alpha, \\ B_j = (t_{k+1}-t_j)^{\alpha+1} - (t_{k+1}-t_{j-1})^{\alpha+1} + (\alpha+1)(t_j-t_{j-1})(t_{k+1}-t_j)^\alpha. \end{cases}$$

After the observation of (2.3), it is obvious to see that this is an implicit scheme. In order to decrease computational complexity, predictor–corrector method is naturally proposed.

The predictor–corrector method can be deduced as the following steps. Firstly, we take equation (2.1) as the predictor item y^P_{k+1} and then we replace y_{k+1} on the right-hand side of scheme (2.3) with y^P_{k+1} to get the corrector item y_{k+1} which contributes to the scheme below.

(iii) The predictor–corrector method can be written as

$$\begin{cases} y^P_{k+1} = \sum\limits_{j=0}^{n-1} \frac{t^j_{k+1}}{j!} y^{(j)}_0 + \sum\limits_{j=0}^{k} w_{j,k+1} f(t_j, y_j), \\ y_{k+1} = \sum\limits_{j=0}^{n-1} \frac{t^j_{k+1}}{j!} y^{(j)}_0 + \Big\{ \sum\limits_{j=0}^{k} \widetilde{w}_{j,k+1} f(t_j, y_j) + \widetilde{w}_{k+1,k+1} f(t_{k+1}, y^P_{k+1}) \Big\}, \end{cases} \tag{2.5}$$

where $w_{j,k+1}$, $\widetilde{w}_{j,k+1}$ are respectively defined by (2.2) and (2.4).

In order to obtain further higher-order numerical schemes on non-uniform meshes, we can also apply other quadrature rules to (1.2), such as Simpson rule, composite trapezoid rule and so on. We omit them here.

3. Stability analysis

In this section, we study the stability analysis for rectangle scheme (2.1), trapezoid scheme (2.3), and predictor–corrector scheme (2.5). Here the stability mainly refers to that if there is a perturbation in the initial condition, then the small change does not cause the large error in the numerical solution [23].

Suppose that y_k and z_k $(k = 1, 2, \cdots, N)$ are two solutions of the rectangle scheme (2.1) with different initial values $y^{(i)}_0$ and $z^{(i)}_0$ $(i = 0, 1, \cdots, n-1)$, respectively. If there exists a positive constant C independent of non-equivalent stepsizes τ_j $(j = 0, 1, \cdots, k)$ and k, such that

$$|y_k - z_k| \le C \sum_{i=0}^{n-1} |y^{(i)}_0 - z^{(i)}_0|, \quad k = 1, 2, \cdots, N,$$

then we say that rectangle scheme (2.1) is stable. It is similar to define the numerical stability for trapezoid scheme (2.3) and predictor–corrector scheme (2.5). Denote C as a generic positive constant that does not depend on meshes or k but on T, α and the smoothness of f if no ambiguousness occurs.

3.1. Several lemmas

Firstly, we introduce some lemmas that will be used in stability analysis as well as error analysis.

Lemma 3.1. *If $\alpha > 0$, k is a nonnegative integer, $\tau_j \le \tau_{j+1}$ $(j = 0, 1, \cdots, k-1)$, then $w_{j,k+1}$ and $\widetilde{w}_{j,k+1}$ defined by equations (2.2) and (2.4) respectively have the following estimates*

$$w_{j,k+1} \le C_\alpha \tau_j (t_{k+1} - t_j)^{\alpha-1}, \quad j = 0, 1, \cdots, k, \tag{3.1}$$

and

$$\widetilde{w}_{j,k+1} \le C_\alpha \begin{cases} \tau_0 t^{\alpha-1}_{k+1}, & \text{if } j = 0, \\ [\tau_j (t_{k+1} - t_j)^{\alpha-1} + \tau_{j-1}(t_{k+1} - t_{j-1})^{\alpha-1}], & \text{if } j = 1, 2, \cdots, k+1, \end{cases} \tag{3.2}$$

where $C_\alpha = \frac{\max\{2, \alpha\}}{\Gamma(\alpha+1)}$.

Proof. It is simple to verify that

$$w_{k,k+1} = \frac{\tau^\alpha_k}{\Gamma(\alpha+1)}$$

For $j = 0, 1, \cdots, k-1$, according to the mean value theorem, there exists a constant $\xi \in (t_j, t_{j+1})$, such that

$$\begin{aligned} \frac{\Gamma(\alpha+1) w_{j,k+1}}{(t_{k+1} - t_j)^{\alpha-1}} &= \frac{(t_{k+1} - t_j)^\alpha - (t_{k+1} - t_{j+1})^\alpha}{(t_{k+1} - t_j)^{\alpha-1}} \\ &= \frac{\alpha \tau_j (t_{k+1} - \xi)^{\alpha-1}}{(t_{k+1} - t_j)^{\alpha-1}} \end{aligned}$$

If $\alpha \geq 1$, then

$$\frac{\Gamma(\alpha+1)w_{j,k+1}}{(t_{k+1}-t_j)^{\alpha-1}} \leq \frac{\alpha\tau_j(t_{k+1}-t_j)^{\alpha-1}}{(t_{k+1}-t_j)^{\alpha-1}} \leq \alpha\tau_j.$$

If $0 < \alpha < 1$, then

$$\begin{aligned}\frac{\Gamma(\alpha+1)w_{j,k+1}}{(t_{k+1}-t_j)^{\alpha-1}} &= \alpha\tau_j(\frac{t_{k+1}-t_j}{t_{k+1}-\xi})^{1-\alpha} \\ &\leq \alpha\tau_j(\frac{t_{k+1}-t_j}{t_{k+1}-t_{j+1}})^{1-\alpha} \\ &= \alpha\tau_j(1+\frac{\tau_j}{\tau_k+\tau_{k+1}+\cdots+\tau_{j+1}})^{1-\alpha} \\ &\leq 2^{1-\alpha}\alpha\tau_j \\ &\leq 2\tau_j.\end{aligned}$$

So inequality (3.1) is proved and inequality (3.2) can be proved in a similar way. All this completes the proof. □

Lemma 3.2 *(Gronwall inequality). (See [41].) Assume that* $\{k_n\}$ *and* $\{p_n\}$ *are nonnegative sequences,* $g_0 \geq 0$*, and the sequence* $\{\phi_n\}$ *satisfies*

$$\begin{cases} \phi_0 \leq g_0, \\ \phi_n \leq g_0 + \sum_{j=0}^{n-1} p_j + \sum_{j=0}^{n-1} k_j\phi_j, \quad n \geq 1. \end{cases}$$

Then

$$\phi_n \leq (g_0 + \sum_{j=0}^{n-1} p_j)\exp(\sum_{j=0}^{n-1} k_j), \quad n \geq 1.$$

Next, we propose a modified Gronwall inequality which is useful to prove the stability and error estimates for the derived numerical methods.

Lemma 3.3. *Assume that* $\alpha,\ C_0,\ T > 0$ *and* $b_{j,k} = C_0\tau_j(t_k-t_j)^{\alpha-1}(j=0,1,\cdots,k-1)$ *for* $0 = t_0 < t_1 < \cdots < t_N = T,\ k = 1,2,\cdots,N$ *where* N *is a positive integer and* $\tau_j = t_{j+1}-t_j$*. Let* g_0 *be positive and the sequence* $\{\psi_k\}$ *meet*

$$\begin{cases} \psi_0 \leq g_0, \\ \psi_k \leq \sum_{j=0}^{k-1} b_{j,k}\psi_j + g_0, \end{cases} \tag{3.3}$$

then

$$\psi_k \leq Cg_0, \quad k = 1,2,\cdots,N. \tag{3.4}$$

Proof. If $\alpha \geq 1$, one gets $b_{j,k} \leq C_0T^{\alpha-1}\tau_j$, that is

$$\psi_k \leq C_0T^{\alpha-1}\sum_{j=0}^{k-1}\tau_j\psi_j + g_0.$$

It naturally leads to (3.4) by using Lemma 3.2.

Next, we study the situation $0 < \alpha < 1$.

Noticing that $\psi_0 \leq g_0$ and using the inequality (3.3) repeatedly, we have

$$\begin{aligned}
\psi_k &\le g_0 + \sum_{j_1=0}^{k-1} b_{j_1,k}\psi_{j_1} \le g_0 + \sum_{j_1=0}^{k-1} b_{j_1,k}\big(g_0 + \sum_{j_2=0}^{j_1-1} b_{j_2,j_1}\psi_{j_2}\big) \\
&= g_0 + g_0\sum_{j_1=0}^{k-1} b_{j_1,k} + \sum_{j_1=0}^{k-1} b_{j_1,k}\sum_{j_2=0}^{j_1-1} b_{j_2,j_1}\psi_{j_2} \\
&\le g_0 + g_0\sum_{j_1=0}^{k-1} b_{j_1,k} + \sum_{j_1=0}^{k-1} b_{j_1,k}\sum_{j_2=0}^{j_1-1} b_{j_2,j_1}\big(g_0 + \sum_{j_3=0}^{j_2-1} b_{j_3,j_2}\psi_{j_3}\big) \\
&= g_0 + g_0\sum_{j_1=0}^{k-1} b_{j_1,k} + g_0\sum_{j_1=0}^{k-1} b_{j_1,k}\sum_{j_2=0}^{j_1-1} b_{j_2,j_1} + \sum_{j_1=0}^{k-1} b_{j_1,k}\sum_{j_2=0}^{j_1-1} b_{j_2,j_1}\sum_{j_3=0}^{j_2-1} b_{j_3,j_2}\psi_{j_3} \\
&\qquad\vdots \\
&\le g_0 + g_0\sum_{j_1=0}^{k-1} b_{j_1,k} + g_0\sum_{j_1=0}^{k-1} b_{j_1,k}\sum_{j_2=0}^{j_1-1} b_{j_2,j_1} + g_0\sum_{j_1=0}^{k-1} b_{j_1,k}\sum_{j_2=0}^{j_1-1} b_{j_2,j_1}\sum_{j_3=0}^{j_2-1} b_{j_3,j_2} \\
&\quad + \cdots + g_0\sum_{j_1=0}^{k-1} b_{j_1,k}\sum_{j_2=0}^{j_1-1} b_{j_2,j_1}\cdots\sum_{j_k=0}^{j_{k-1}-1} b_{j_k,j_{k-1}} \\
&= g_0 + g_0\sum_{j_1=0}^{k-1} b_{j_1,k} + g_0\sum_{j_2=0}^{k-1} b_{j_2,k}\sum_{j_1=0}^{j_2-1} b_{j_1,j_2} + g_0\sum_{j_3=0}^{k-1} b_{j_3,k}\sum_{j_2=0}^{j_3-1} b_{j_2,j_3}\sum_{j_1=0}^{j_2-1} b_{j_1,j_2} \\
&\quad + \cdots + g_0\sum_{j_k=0}^{k-1} b_{j_k,k}\sum_{j_{k-2}=0}^{j_{k-1}-1} b_{j_{k-2},j_{k-1}}\cdots\sum_{j_1=0}^{j_2-1} b_{j_1,j_2}.
\end{aligned} \tag{3.5}$$

In the following, we show that the right-hand side of the last equality of (3.5) is bounded. Since $f(x) = x^\beta(t_{j_s} - x)^{\alpha-1}(0 < x < t_{j_s}, \beta \ge 0)$ is monotonically increasing, one has

$$\begin{aligned}
\sum_{j_r=0}^{j_s-1} b_{j_r,j_s} t_{j_r}^\beta &\triangleq C_0\sum_{j_r=0}^{j_s-1}\tau_{j_r}(t_{j_s}-t_{j_r})^{\alpha-1}t_{j_r}^\beta, \quad r<s \\
&= C_0\sum_{j_r=0}^{j_s-1}\tau_{j_r} f(t_{j_r}) \\
&\le C_0\int_0^{t_{j_s}} f(t)dt \\
&= C_0 t_{j_s}^{\alpha+\beta}B(\alpha,\beta+1),
\end{aligned}$$

where $B(\cdot,\cdot)$ denotes Euler beta function defined by

$$B(w,z) = \int_0^1 (1-t)^{w-1}t^{z-1}dt, \quad z, w > 0.$$

Hence, we have

$$\begin{aligned}
F_{r,k} &= g_0 \sum_{j_r=0}^{k-1} b_{j_r,k} \sum_{j_{r-1}=0}^{j_r-1} b_{j_{r-1},j_r} \cdots \sum_{j_1=0}^{j_2-1} b_{j_1,j_2} \\
&\le g_0 C_0 \sum_{j_r=0}^{k-1} b_{j_r,k} \sum_{j_{r-1}=0}^{j_r-1} b_{j_{r-1},j_r} \cdots \sum_{j_2=0}^{j_3-1} b_{j_2,j_3} t_{j_2}^{\alpha} B(\alpha,1) \\
&\le g_0 C_0^2 \sum_{j_r=0}^{k-1} b_{j_r,k} \sum_{j_{r-1}=0}^{j_r-1} b_{j_{r-1},j_r} \cdots \sum_{j_3=0}^{j_4-1} b_{j_3,j_4} t_{j_3}^{2\alpha} B(\alpha,1) B(\alpha,\alpha+1) \\
&\ \ \vdots \\
&\le g_0 C_0^r t_k^{r\alpha} \prod_{j=0}^{r-1} B(\alpha, j\alpha+1) \\
&\le g_0 C_0^r T^{r\alpha} \prod_{j=0}^{r-1} B(\alpha, j\alpha+1) \\
&= g_0 C_0^r T^{r\alpha} \cdot \frac{\Gamma(\alpha)}{\Gamma(\alpha+1)} \cdot \frac{\Gamma(\alpha)\Gamma(\alpha+1)}{\Gamma(2\alpha+1)} \cdot \cdots \cdot \\
&\quad \frac{\Gamma(\alpha)\Gamma((r-2)\alpha+1)}{\Gamma((r-1)\alpha+1)} \cdot \frac{\Gamma(\alpha)\Gamma((r-1)\alpha+1)}{\Gamma(r\alpha+1)} \\
&= g_0 \frac{C_0^r T^{r\alpha} (\Gamma(\alpha))^r}{\Gamma(r\alpha+1)},
\end{aligned} \tag{3.6}$$

where $r = 1, 2, \cdots, k$.

Denote $a_r = \frac{C_0^r T^{r\alpha}(\Gamma(\alpha))^r}{\Gamma(r\alpha+1)}$, $r = 1, 2, \ldots, k$. Then

$$\frac{a_{r+1}}{a_r} = C_0 T^{\alpha} \frac{\Gamma(\alpha)\Gamma(r\alpha+1)}{\Gamma((r+1)\alpha+1)}.$$

According to Stirling's formula in terms of the Euler gamma function, for large z one gets

$$\Gamma(z) = \sqrt{2\pi} z^{z-\frac{1}{2}} e^{-z} e^{\frac{\theta_z}{12z}}, \ \theta_z \in (0,1).$$

So, for large r one has

$$\begin{aligned}
\frac{\Gamma(r\alpha+1)}{\Gamma((r+1)\alpha+1)} &= \frac{r\Gamma(r\alpha)}{(r+1)\Gamma((r+1)\alpha)} \\
&= \left(\frac{r}{r+1}\right)^{r\alpha+\frac{1}{2}} \left(\frac{e}{\alpha}\right)^{\alpha} (r+1)^{-\alpha} \exp\left(\frac{\theta_r}{12r\alpha} - \frac{\theta_{r+1}}{12(r+1)\alpha}\right) \\
&\le C_1 (r+1)^{-\alpha},
\end{aligned}$$

where $\theta_r, \theta_{r+1} \in (0,1)$, $C_1 = \left(\frac{e}{\alpha}\right)^{\alpha} e^{\frac{1}{12\alpha}}$.

Thus,

$$0 \le \lim_{r\to\infty} \frac{a_{r+1}}{a_r} \le \lim_{r\to\infty} C_0 C_1 T^{\alpha} \Gamma(\alpha)(r+1)^{-\alpha} = 0,$$

i.e.,

$$\lim_{r\to\infty} \frac{a_{r+1}}{a_r} = 0,$$

which means that $\sum_{r=1}^{\infty} a_r$ is convergent. Hence, the right-hand side of the last equality of (3.5) is bounded, that is

$$\psi_k \le g_0 + g_0 \sum_{r=1}^{k} a_r \le C g_0, \tag{3.7}$$

which shows that (3.4) holds for the case $0 < \alpha < 1$. It follows that (3.4) is true for all $\alpha > 0$. □

The above lemma is very useful for numerical stability analysis and error estimate. There have existed some other kinds of Gronwall inequalities, for example, see [14,39,41].

3.2. Stability analysis

In this section, we first give the stability analysis of rectangle scheme (2.1) in the following theorem.

Theorem 3.1. *Suppose that $y_j\,(j=1,2,\cdots,k)$ are the solutions of the rectangle scheme (2.1) where the non-equidistant stepsize is non-decreasing, $f(t,y)$ satisfies the Lipschitz condition with respect to the second argument y with a Lipschitz constant L on the existed interval of its unique solution. Then the rectangle scheme (2.1) is stable.*

Proof. Suppose $\widetilde{y}_0^{(i)}$ $(i=0,1,\cdots,n-1)$ and $\widetilde{y}_j$ $(j=0,1,\cdots,k+1)$ are the perturbations of $y_0^{(i)}$ and y_j, respectively. It follows from (2.1) that

$$\begin{cases} y_{k+1}=\displaystyle\sum_{j=0}^{n-1}\frac{t_{k+1}^j}{j!}y_0^{(j)}+\sum_{j=0}^{k}w_{j,k+1}f(t_j,y_j),\\ y_{k+1}+\widetilde{y}_{k+1}=\displaystyle\sum_{j=0}^{n-1}\frac{t_{k+1}^j}{j!}(y_0^{(j)}+\widetilde{y}_0^{(j)})+\sum_{j=0}^{k}w_{j,k+1}f(t_j,y_j+\widetilde{y}_j).\end{cases}$$

So we obtain the following perturbation equation

$$\widetilde{y}_{k+1}=\sum_{j=0}^{n-1}\frac{t_{k+1}^j}{j!}\widetilde{y}_0^{(j)}+\sum_{j=0}^{k}w_{j,k+1}\Big(f(t_j,y_j+\widetilde{y}_j)-f(t_j,y_j)\Big).$$

Denoting by $\eta_0=\max\limits_{0\le j\le n-1}|\widetilde{y}_0^{(j)}|$, we get

$$\begin{aligned}|\widetilde{y}_{k+1}|&=\Big|\sum_{j=0}^{n-1}\frac{t_{k+1}^j}{j!}\widetilde{y}_0^{(j)}+\sum_{j=0}^{k}w_{j,k+1}\Big(f(t_j,y_j+\widetilde{y}_j)-f(t_j,y_j)\Big)\Big|\\ &\le\sum_{j=0}^{n-1}\frac{t_{k+1}^j}{j!}\eta_0+L\sum_{j=0}^{k}w_{j,k+1}|\widetilde{y}_j|.\end{aligned}$$

Using Lemmas 3.1 and 3.3, we reach that

$$\begin{aligned}|\widetilde{y}_{k+1}|&\le C_1\Big(\sum_{j=0}^{n-1}\frac{t_{k+1}^j}{j!}\eta_0\Big)\\ &\le C_1\Big(\sum_{j=0}^{n-1}\frac{T^j}{j!}\Big)\eta_0\\ &=C\eta_0,\end{aligned}$$

where $C=C_1\big(\sum\limits_{j=0}^{n-1}\frac{T^j}{j!}\big)$. The proof is thus ended. □

Similar to the proof of Theorem 3.1, we have the following theorem whose proof is omitted here or left to the interested readers.

Theorem 3.2. *Suppose that $y_j\,(j=1,2,\cdots,k)$ are the solutions of the trapezoid scheme (2.3) where the non-equidistant stepsize is non-decreasing, $f(t,y)$ satisfies the Lipschitz condition with respect to the second argument y with a Lipschitz constant L on the existed interval of its unique solution. Then the trapezoid scheme (2.3) is stable.*

Next, we present the stability of scheme (2.5).

Theorem 3.3. *Suppose that $y_j\,(j=1,2,\cdots,k)$ are the solutions of the predictor–corrector scheme (2.5) where the non-equidistant stepsize is non-decreasing, $f(t,y)$ satisfies the Lipschitz condition with respect to the second argument y with a Lipschitz constant L on the existed interval of its unique solution. Then the predictor–corrector scheme (2.5) is stable.*

Proof. Suppose $\widetilde{y}_0^{(i)}$ $(i=0,1,\cdots,n-1)$, $\widetilde{y}_j$ $(j=0,1,\cdots,k+1)$ and $\widetilde{y}_{k+1}^P$ $(k=0,1,\cdots,N-1)$ are the perturbations of $y_0^{(i)}$, y_j and y_{k+1}^P, respectively. Then we get the following perturbation equations

$$\begin{cases} \widetilde{y}_{k+1}^P = \sum_{j=0}^{n-1} \frac{t_{k+1}^j}{j!} \widetilde{y}_0^{(j)} + \sum_{j=0}^{k} w_{j,k+1}\Big(f(t_j, y_j+\widetilde{y}_j) - f(t_j, y_j)\Big), \\ \widetilde{y}_{k+1} = \sum_{j=0}^{n-1} \frac{t_{k+1}^j}{j!} \widetilde{y}_0^{(j)} + \Big\{ \sum_{j=0}^{k} \widetilde{w}_{j,k+1}\Big(f(t_j, y_j+\widetilde{y}_j) - f(t_j, y_j)\Big) \\ \qquad + \widetilde{w}_{k+1,k+1}\Big(f(t_{k+1}, y_{k+1}^P + \widetilde{y}_{k+1}^P) - f(t_{k+1}, y_{k+1}^P)\Big)\Big\}. \end{cases}$$

Denote by $\eta_0 = \max\limits_{0\le j\le n-1} |\widetilde{y}_0^{(j)}|$. Noticing that the following estimate holds

$$\widetilde{w}_{k+1,k+1} \le \frac{T^\alpha}{\Gamma(\alpha+2)}, \tag{3.8}$$

one has

$$\begin{aligned} |\widetilde{y}_{k+1}| &= \Big| \sum_{j=0}^{n-1} \frac{t_{k+1}^j}{j!} \widetilde{y}_0^{(j)} + \Big\{ \sum_{j=0}^{k} \widetilde{w}_{j,k+1}\Big(f(t_j, y_j+\widetilde{y}_j) - f(t_j, y_j)\Big) \\ &\quad + \widetilde{w}_{k+1,k+1}\Big(f(t_{k+1}, y_{k+1}^P + \widetilde{y}_{k+1}^P) - f(t_{k+1}, y_{k+1}^P)\Big)\Big\}\Big| \\ &\le \sum_{j=0}^{n-1} \frac{t_{k+1}^j}{j!} |\widetilde{y}_0^{(j)}| + L\sum_{j=0}^{k} \widetilde{w}_{j,k+1}|\widetilde{y}_j| + L\widetilde{w}_{k+1,k+1}|\widetilde{y}_{k+1}^P| \\ &\le \eta_0 \sum_{j=0}^{n-1} \frac{T^j}{j!} + L\sum_{j=0}^{k} \widetilde{w}_{j,k+1}|\widetilde{y}_j| + \frac{LT^\alpha}{\Gamma(\alpha+2)}|\widetilde{y}_{k+1}^P| \\ &\le \Big(1+\frac{LT^\alpha}{\Gamma(2+\alpha)}\Big)\eta_0 \sum_{j=0}^{n-1} \frac{T^j}{j!} + L\sum_{j=0}^{k}\Big(\widetilde{w}_{j,k+1} + \frac{LT^\alpha}{\Gamma(\alpha+2)} w_{j,k+1}\Big)|\widetilde{y}_j|. \end{aligned}$$

According to Lemmas 3.1 and 3.3, it leads to

$$\begin{aligned} |\widetilde{y}_{k+1}| &\le \Big(1+\frac{LT^\alpha}{\Gamma(2+\alpha)}\Big)\eta_0 \sum_{j=0}^{n-1} \frac{T^j}{j!} + LC_\alpha\Big(2+\frac{LT^\alpha}{\Gamma(\alpha+2)}\Big)\sum_{j=0}^{k} \tau_j (t_{k+1}-t_j)^{\alpha-1}|\widetilde{y}_j| \\ &\le C_1\Big(1+\frac{LT^\alpha}{\Gamma(2+\alpha)}\Big)\eta_0 \sum_{j=0}^{n-1} \frac{T^j}{j!} \\ &= C\eta_0, \end{aligned}$$

where $C = C_1\Big(1+\frac{LT^\alpha}{\Gamma(2+\alpha)}\Big)\sum\limits_{j=0}^{n-1} \frac{T^j}{j!}$. All this ends the proof. □

Remark 3.1. The global Lipschitz conditions in Theorems 3.1–3.3 can be weakened to the local Lipschitz conditions. The proof techniques are almost the same as those studied in [23].

4. Error analysis

In this section, we give the error analysis of rectangle scheme (2.1), trapezoid scheme (2.3), and predictor–corrector scheme (2.5).

At first, we present some lemmas that will be used later on.

Lemma 4.1. *If $g(t) \in C^1[0, T]$, then*

$$\left|\frac{1}{\Gamma(\alpha)}\int_0^{t_{k+1}}(t_{k+1}-s)^{\alpha-1}g(s)ds-\sum_{j=0}^{k}w_{j,k+1}g(t_j)\right| \le \frac{\|g'\|_\infty}{\Gamma(\alpha+1)}T^\alpha\tau_{max}.$$

Proof.

$$\begin{aligned}
&\Big|\frac{1}{\Gamma(\alpha)}\int_0^{t_{k+1}}(t_{k+1}-s)^{\alpha-1}g(s)ds-\sum_{j=0}^{k}w_{j,k+1}g(t_j)\Big| \\
&=\frac{1}{\Gamma(\alpha)}\left|\sum_{j=0}^{k}\int_{t_j}^{t_{j+1}}(t_{k+1}-s)^{\alpha-1}\Big(g(t_j)-g(s)\Big)ds\right| \\
&\le \frac{1}{\Gamma(\alpha)}\sum_{j=0}^{k}\int_{t_j}^{t_{j+1}}(t_{k+1}-s)^{\alpha-1}\left|(s-t_j)g'(\xi_j)\right|ds, \quad \xi_j\in(t_j,t_{j+1}) \\
&\le \frac{1}{\Gamma(\alpha)}\|g'\|_\infty\tau_{max}\sum_{j=0}^{k}\int_{t_j}^{t_{j+1}}(t_{k+1}-s)^{\alpha-1}ds \\
&=\frac{1}{\Gamma(\alpha+1)}\|g'\|_\infty t_{k+1}^\alpha\tau_{max} \le \frac{\|g'\|_\infty}{\Gamma(\alpha+1)}T^\alpha\tau_{max}.
\end{aligned}$$

This finishes the proof. □

Lemma 4.2. *If $g(t) \in C^2[0, T]$, then*

$$\left|\frac{1}{\Gamma(\alpha)}\int_0^{t_{k+1}}(t_{k+1}-s)^{\alpha-1}g(s)ds-\sum_{j=0}^{k+1}\widetilde{w}_{j,k+1}g(t_j)\right| \le \frac{\|g''\|_\infty}{2\Gamma(\alpha+1)}T^\alpha\tau_{max}^2.$$

Proof.

$$\begin{aligned}
&\Big|\frac{1}{\Gamma(\alpha)}\int_0^{t_{k+1}}(t_{k+1}-s)^{\alpha-1}g(s)ds-\sum_{j=0}^{k+1}\widetilde{w}_{j,k+1}g(t_j)\Big| \\
&=\frac{1}{\Gamma(\alpha)}\left|\sum_{j=0}^{k}\int_{t_j}^{t_{j+1}}(t_{k+1}-s)^{\alpha-1}\left\{g(s)-\frac{s-t_{j+1}}{t_j-t_{j+1}}g(t_j)-\frac{s-t_j}{t_{j+1}-t_j}g(t_{j+1})\right\}ds\right| \\
&\le \frac{1}{\Gamma(\alpha)}\sum_{j=0}^{k}\int_{t_j}^{t_{j+1}}(t_{k+1}-s)^{\alpha-1}\left|\frac{1}{2}(s-t_{j+1})(s-t_j)g''(\xi_j)\right|ds, \quad \xi_j\in(t_j,t_{j+1}) \\
&\le \frac{\|g''\|_\infty}{2\Gamma(\alpha)}\tau_{max}^2\sum_{j=0}^{k}\int_{t_j}^{t_{j+1}}(t_{k+1}-s)^{\alpha-1}ds \\
&=\frac{\|g''\|_\infty}{2\Gamma(\alpha+1)}t_{k+1}^\alpha\tau_{max}^2 \le \frac{\|g''\|_\infty}{2\Gamma(\alpha+1)}T^\alpha\tau_{max}^2.
\end{aligned}$$

The proof is thus completed. □

In the above two lemmas, the smooth conditions of the integrand $g(t)$ are sufficient ones which are somewhat strong. In such situations, non-uniform meshes seem not to have superiority. It means that the convergent rates of both non-uniform meshes and uniform ones are highly possible the same for smooth functions. But for non-smooth functions, non-uniform meshes are much more suitable than uniform meshes. Now we give another lemma based on concrete non-uniform meshes to show this.

Lemma 4.3. *Let $g(t) = t^\sigma$ $(0 < \sigma < 1)$ which is not smooth at the origin $t = 0$. Set non-uniform meshes with variable stepsizes $\tau_j = t_{j+1} - t_j = (j+1)\mu,\ 0 \le j \le N-1$, where $\mu = \frac{2T}{N(N+1)}$. Then for $0 < \alpha < 1$, we have*

$$\begin{aligned}&\left|\frac{1}{\Gamma(\alpha)}\int_0^{t_{k+1}}(t_{k+1}-s)^{\alpha-1}g(s)ds - \sum_{j=0}^{k+1}\widetilde{w}_{j,k+1}g(t_j)\right|\\ &\le C_{\sigma,\alpha}N^{-2(\sigma+\alpha)}(k+1)^{2(\sigma+\alpha-1)},\end{aligned} \tag{4.1}$$

where $k = 0, 1, \cdots, N-1$, and $C_{\sigma,\alpha}$ is a constant depending only on T, α, σ.

Proof.

$$\begin{aligned}&\left|\frac{1}{\Gamma(\alpha)}\int_0^{t_{k+1}}(t_{k+1}-s)^{\alpha-1}g(s)ds - \sum_{j=0}^{k+1}\widetilde{w}_{j,k+1}g(t_j)\right|\\ &= \frac{1}{\Gamma(\alpha)}\left|\sum_{j=0}^{k}\int_{t_j}^{t_{j+1}}(t_{k+1}-s)^{\alpha-1}\left\{g(s) - \frac{s-t_{j+1}}{t_j-t_{j+1}}g(t_j) - \frac{s-t_j}{t_{j+1}-t_j}g(t_{j+1})\right\}ds\right|\\ &\le \frac{1}{\Gamma(\alpha)}\sum_{j=1}^{k}\int_{t_j}^{t_{j+1}}(t_{k+1}-s)^{\alpha-1}\left|\frac{1}{2}(s-t_{j+1})(s-t_j)\sigma(\sigma-1)\xi_j^{\sigma-2}\right|ds\\ &\quad + \frac{1}{\Gamma(\alpha)}\int_0^{t_1}(t_{k+1}-s)^{\alpha-1}(s^\sigma - st_1^{\sigma-1})ds\\ &= I_1 + I_2,\end{aligned} \tag{4.2}$$

where $\xi_j \in (t_j, t_{j+1})$. It is easy to show that the above inequality is bounded by $C_{\sigma,\alpha}t_1^{\sigma+\alpha}$ when $k = 0$.

Next we focus on the case $k \ge 1$. For the first part I_1 on the right-hand side of (4.1), we have the following estimate.

$$\begin{aligned}I_1 &\le \frac{1}{2\Gamma(\alpha)}\sigma(1-\sigma)\sum_{j=1}^{k}\tau_j^2 t_j^{\sigma-2}\int_{t_j}^{t_{j+1}}(t_{k+1}-s)^{\alpha-1}ds\\ &\le \frac{1}{2\Gamma(\alpha)}\sigma(1-\sigma)\sum_{j=1}^{k}\tau_j^3 t_j^{\sigma-2}(t_{k+1}-t_{j+1})^{\alpha-1}\\ &\le C_{\sigma,\alpha}\left(\frac{T}{N(N+1)}\right)^{\sigma+\alpha}\sum_{j=1}^{k}(j+1)^{\sigma+1}j^{\sigma-2}(k+j+3)^{\alpha-1}(k-j)^{\alpha-1}\\ &\le C_{\sigma,\alpha}\left(\frac{T}{N(N+1)}\right)^{\sigma+\alpha}\sum_{j=1}^{k}\frac{j+1}{j}(j+1)^{\sigma}j^{\sigma-1}(k+j+3)^{\alpha-1}(k-j)^{\alpha-1}\\ &\le C_{\sigma,\alpha}\left(\frac{T}{N(N+1)}\right)^{\sigma+\alpha}(k+1)^{\sigma+\alpha-1}\sum_{j=1}^{k}j^{\sigma-1}(k-j)^{\alpha-1}\\ &\le C_{\sigma,\alpha}\left(\frac{T}{N(N+1)}\right)^{\sigma+\alpha}(k+1)^{\sigma+\alpha-1}\int_0^{k}s^{\sigma-1}(k-s)^{\alpha-1}ds\\ &\le C_{\sigma,\alpha}\left(\frac{T}{N(N+1)}\right)^{\sigma+\alpha}(k(k+1))^{\sigma+\alpha-1}.\end{aligned}$$

For $0 < \sigma + \alpha < 1$, we get that

$$\begin{aligned} I_1 &\le C_{\sigma,\alpha} N^{-2(\sigma+\alpha)} k^{2(\sigma+\alpha-1)} \\ &\le C_{\sigma,\alpha} N^{-2(\sigma+\alpha)} (k+1)^{2(\sigma+\alpha-1)} \left(1-\frac{1}{k+1}\right)^{2(\sigma+\alpha-1)} \\ &\le 4C_{\sigma,\alpha} N^{-2(\sigma+\alpha)} (k+1)^{2(\sigma+\alpha-1)}. \end{aligned} \tag{4.3}$$

Or else $1 \le \sigma + \alpha < 2$, then $I_1 \le C_{\sigma,\alpha} N^{-2(\sigma+\alpha)} (k+1)^{2(\sigma+\alpha-1)}$. Thus we can reach that $I_1 \le C_{\sigma,\alpha} N^{-2(\sigma+\alpha)} (k+1)^{2(\sigma+\alpha-1)}$ with a generic constant for arbitrary $0 < \alpha, \sigma < 1$.

For the second part I_2, one has

$$\begin{aligned} I_2 &= \frac{1}{\Gamma(\alpha)} \int_0^{t_1} (t_{k+1}-s)^{\alpha-1} (s^\sigma - s t_1^{\sigma-1}) ds \\ &\le \frac{1}{\Gamma(\alpha)} \left(\frac{1}{\sigma+1} - \frac{1}{2}\right) t_1^{\sigma+1} (t_{k+1}-t_1)^{\alpha-1} \\ &\le C_{\sigma,\alpha} \left(\frac{T}{N(N+1)}\right)^{\sigma+\alpha} (k(k+1))^{\alpha-1} \\ &\le 4C_{\sigma,\alpha} N^{-2(\sigma+\alpha)} (k+1)^{2(\alpha-1)}, \end{aligned}$$

where the technique of deriving (4.3) is used in the last inequality.

So, combining I_1 with I_2, we can obtain inequality (4.1). The proof is complete. □

Remark 4.1. For uniform meshes with a uniform stepsize $h = \frac{T}{N}$ and $g(t) = t^\sigma$ $(0 < \sigma < 1)$, it has been proved in [8] that

$$\left| \frac{1}{\Gamma(\alpha)} \int_0^{t_{k+1}} (t_{k+1}-s)^{\alpha-1} g(s) ds - \sum_{j=0}^{k+1} \widetilde{w}_{j,k+1} g(t_j) \right| \le C_{\sigma,\alpha} t_{k+1}^{\alpha-1} h^{\sigma+1},$$

i.e.,

$$\left| \frac{1}{\Gamma(\alpha)} \int_0^{t_{k+1}} (t_{k+1}-s)^{\alpha-1} g(s) ds - \sum_{j=0}^{k+1} \widetilde{w}_{j,k+1} g(t_j) \right| \le C_{\sigma,\alpha} N^{-(\sigma+\alpha)} (k+1)^{\alpha-1}. \tag{4.4}$$

In view of inequalities (4.1) and (4.4), we find that the order of convergence with non-uniform meshes are higher than that with uniform meshes.

From the above Lemmas 4.1 and 4.2, one can easily get the following theorems.

Theorem 4.1. *If ${}_C D_{0,t}^\alpha y(t) \in C^1[0,T]$ and the non-equidistant stepsize is non-decreasing, then the rectangle scheme (2.1) for equation (1.2) has the following estimate,*

$$|y_{k+1} - y(t_{k+1})| \le C\tau_{max}, \quad k = 0, 1, \ldots, N-1. \tag{4.5}$$

Theorem 4.2. *If ${}_C D_{0,t}^\alpha y(t) \in C^2[0,T]$ and the non-equidistant stepsize is non-decreasing, then the trapezoid scheme (2.3) for equation (1.2) has the following estimate,*

$$|y_{k+1} - y(t_{k+1})| \le C\tau_{max}^2, \quad k = 0, 1, \ldots, N-1. \tag{4.6}$$

Next, we derive the error estimate of scheme (2.5).

Theorem 4.3. *If ${}_C D_{0,t}^\alpha y(t) \in C^2[0,T]$ and the non-equidistant stepsize is non-decreasing, then the predictor–corrector scheme (2.5) has the estimate below,*

$$|y_{k+1} - y(t_{k+1})| \le C\tau_{max}^q, \quad k = 0, 1, \ldots, N-1, \tag{4.7}$$

where $q = \min\{2, 1+\alpha\}$.

Proof. Since $f(t, y(t)) = {}_C D_{0,t}^\alpha y(t) \in C^2[0,T]$ is bounded, then there exists a constant $M > 0$, such that $|f_t| \le M$ and $|f_{tt}| \le M$. We denote $e_{k+1} = |y(t_{k+1}) - y_{k+1}|$ for $k = 0, 1, \ldots, N-1$. By tedious calculations, one has

$$\begin{aligned} e_{k+1} &= \Big| \frac{1}{\Gamma(\alpha)} \int_0^{t_{k+1}} (t_{k+1}-s)^{\alpha-1} f(s,y(s))ds \\ &\quad - \Big\{ \sum_{j=0}^{k} \widetilde{w}_{j,k+1} f(t_j,y_j) + \widetilde{w}_{k+1,k+1} f(t_{k+1},y^P_{k+1}) \Big\} \Big| \\ &\le \Big| \frac{1}{\Gamma(\alpha)} \int_0^{t_{k+1}} (t_{k+1}-s)^{\alpha-1} f(s,y(s))ds - \sum_{j=0}^{k+1} \widetilde{w}_{j,k+1} f(t_j,y(t_j)) \Big| \\ &\quad + \Big| \sum_{j=0}^{k+1} \widetilde{w}_{j,k+1} f(t_j,y(t_j)) - \Big\{ \sum_{j=0}^{k} \widetilde{w}_{j,k+1} f(t_j,y_j) \\ &\quad + \widetilde{w}_{k+1,k+1} f(t_{k+1},y^P_{k+1}) \Big\} \Big| \\ &\le \Big| \frac{1}{\Gamma(\alpha)} \int_0^{t_{k+1}} (t_{k+1}-s)^{\alpha-1} f(s,y(s))ds - \sum_{j=0}^{k+1} \widetilde{w}_{j,k+1} f(t_j,y(t_j)) \Big| \\ &\quad + \Big| \sum_{j=0}^{k} \widetilde{w}_{j,k+1} \Big[f(t_j,y(t_j)) - f(t_j,y_j) \Big] \Big| \\ &\quad + \Big| \widetilde{w}_{k+1,k+1} \Big[f(t_{k+1},y(t_{k+1})) - f(t_{k+1},y^P_{k+1}) \Big] \Big| \\ &\triangleq I_1 + I_2 + I_3. \end{aligned}$$

It follows from Lemma 4.2 that

$$I_1 \le C_1 \tau_{max}^2,$$

where $C_1 = \frac{MT^\alpha}{2\Gamma(1+\alpha)}$.

Since f is presumed to satisfy the Lipschitz condition with a Lipschitz constant L, one has

$$\begin{aligned} I_2 &= \Big| \sum_{j=0}^{k} \widetilde{w}_{j,k+1} \big\{ f(t_j,y(t_j)) - f(t_j,y_j) \big\} \Big| \\ &\le \sum_{j=0}^{k} \widetilde{w}_{j,k+1} \big| f(t_j,y(t_j)) - f(t_j,y_j) \big| \\ &\le L \sum_{j=0}^{k} \widetilde{w}_{j,k+1} |y(t_j) - y_j| \\ &= L \sum_{j=0}^{k} \widetilde{w}_{j,k+1} e_j. \end{aligned}$$

By the definition of $\widetilde{w}_{k+1,k+1}$, we get

$$\widetilde{w}_{k+1,k+1} = \frac{1}{\Gamma(\alpha+2)} (t_{k+1}-t_k)^\alpha = \frac{1}{\Gamma(\alpha+2)} \tau_k^\alpha \le \frac{1}{\Gamma(\alpha+2)} \tau_{max}^\alpha.$$

Thus

$$\begin{aligned} I_3 &= \Big| \widetilde{w}_{k+1,k+1} \Big\{ f(t_{k+1},y(t_{k+1})) - f(t_{k+1},y^P_{k+1}) \Big\} \Big| \\ &\le \frac{1}{\Gamma(\alpha+2)} \tau_{max}^\alpha \Big| f(t_{k+1},y(t_{k+1})) - f(t_{k+1},y^P_{k+1}) \Big| \\ &\le \frac{L}{\Gamma(\alpha+2)} \tau_{max}^\alpha \Big| y(t_{k+1}) - y^P_{k+1} \Big| \end{aligned}$$

$$
\begin{aligned}
&= \frac{L\tau_{max}^{\alpha}}{\Gamma(\alpha+2)} \cdot \Big| \frac{1}{\Gamma(\alpha)} \int_0^{t_{k+1}} (t_{k+1}-s)^{\alpha-1} f(s, y(s))ds - \sum_{j=0}^{k} w_{j,k+1} f(t_j, y_j) \Big| \\
&\le \frac{L\tau_{max}^{\alpha}}{\Gamma(\alpha+2)} \Big| \frac{1}{\Gamma(\alpha)} \int_0^{t_{k+1}} (t_{k+1}-s)^{\alpha-1} {}_C D_{0,s}^{\alpha} y(s)ds - \sum_{j=0}^{k} w_{j,k+1} {}_C D_{0,t_j}^{\alpha} y(t_j) \Big| \\
&+ \frac{L\tau_{max}^{\alpha}}{\Gamma(\alpha+2)} \Big| \sum_{j=0}^{k} w_{j,k+1} {}_C D_{0,t_j}^{\alpha} y(t_j) - \sum_{j=0}^{k} w_{j,k+1} f(t_j, y_j) \Big| \\
&\le \frac{L\tau_{max}^{\alpha}}{\Gamma(\alpha+2)} \Big| \frac{1}{\Gamma(\alpha)} \int_0^{t_{k+1}} (t_{k+1}-s)^{\alpha-1} {}_C D_{0,s}^{\alpha} y(s)ds - \sum_{j=0}^{k} w_{j,k+1} {}_C D_{0,t_j}^{\alpha} y(t_j) \Big| \\
&+ \frac{L\tau_{max}^{\alpha}}{\Gamma(\alpha+2)} \sum_{j=0}^{k} w_{j,k+1} \big| f(t_j, y(t_j)) - f(t_j, y_j) \big| \\
&\le C_2 \tau_{max}^{\alpha+1} + \frac{L^2 T^{\alpha}}{\Gamma(\alpha+2)} \sum_{j=0}^{k} w_{j,k+1} e_j,
\end{aligned}
$$

where Lemma 4.1 is utilized and $C_2 = \frac{LMT^{\alpha}}{\Gamma(\alpha+1)\Gamma(\alpha+2)}$.

Therefore,

$$
\begin{aligned}
e_{k+1} &\le I_1 + I_2 + I_3 \\
&\le C_1 \tau_{max}^2 + C_2 \tau_{max}^{\alpha+1} + L \sum_{j=0}^{k} \widetilde{w}_{j,k+1} e_j + \frac{L^2 T^{\alpha}}{\Gamma(\alpha+2)} \sum_{j=0}^{k} w_{j,k+1} e_j \\
&= C_1 \tau_{max}^2 + C_2 \tau_{max}^{\alpha+1} + L \sum_{j=0}^{k} \Big\{ \widetilde{w}_{j,k+1} + \frac{LT^{\alpha}}{\Gamma(\alpha+2)} w_{j,k+1} \Big\} e_j.
\end{aligned}
$$

Then, it follows from Lemmas 3.1 and 3.3 again that

$$
\begin{aligned}
e_{k+1} &\le C_3 (C_1 \tau_{max}^2 + C_2 \tau_{max}^{\alpha+1}) \\
&\le C \tau_{max}^{q},
\end{aligned}
$$

where $q = \min\{2,\ 1+\alpha\}$. The proof is thus complete. □

Remark 4.2. The smooth condition ${}_C D_{0,t}^{\alpha} y(t) \in C^2[0, T]$ is a sufficient one. For smooth functions in $[0, T]$, the convergent rate of non-uniform meshes and uniform ones are almost the same. For non-smooth functions in $[0, T]$ (i.e. whose derivative(s) may not exist at $t = 0$ or other points), however, the non-uniform meshes behave better than the uniform meshes.

Remark 4.3. The conclusions in Theorems 4.1–4.3 still hold if the global Lipschitz conditions are reduced to the local Lipschitz ones [23].

5. Numerical examples

In this section, we present four numerical examples as follows.

If $\tau_j = \text{const.}$, $j = 0, 1, \cdots, N-1$, that is, equidistant division on the interval $[0, T]$, then the convergence rate of predictor–corrector scheme reduced to $2-\alpha$ for $\alpha \in [1/2, 1)$ when $y(t)$ satisfies $y(t) \in C^2[0, T]$. This drawback will be overcome when the non-uniform meshes are used.

We adopt the non-uniform meshes defined as

$$\tau_k = (k+1)\mu, \quad 0 \le k \le N-1,$$

where $\mu = \frac{2T}{N(N+1)}$. In the following, we always use these non-uniform meshes to solve the fractional differential equations (see Examples 5.1–5.3).

Now, we give an example to test the error estimates.

Table 1
Absolute errors at $t = 1$ for equation (5.1) using rectangle scheme (2.1).

N	$\alpha = 0.25$	EOC	$\alpha = 0.75$	EOC	$\alpha = 1.25$	EOC	$\alpha = 1.75$	EOC
80	8.64E−02		6.40E−02		4.99E−02		3.63E−02	
160	3.95E−02	1.13	3.15E−02	1.02	2.53E−02	0.98	1.85E−02	0.97
320	1.85E−02	1.09	1.56E−02	1.02	1.28E−02	0.99	9.35E−03	0.99
640	8.85E−03	1.07	7.72E−03	1.01	6.41E−03	0.99	4.70E−03	0.99
1280	4.26E−03	1.05	3.84E−03	1.01	3.21E−03	1.00	2.36E−03	1.00
2560	2.06E−03	1.05	1.91E−03	1.00	1.61E−03	1.00	1.18E−03	1.00

Table 2
Absolute errors at $t = 1$ for equation (5.1) using trapezoid scheme (2.3).

N	$\alpha = 0.25$	EOC	$\alpha = 0.75$	EOC	$\alpha = 1.25$	EOC	$\alpha = 1.75$	EOC
80	7.95E−04		1.15E−03		8.96E−04		5.73E−04	
160	2.14E−04	1.89	2.90E−04	1.98	2.25E−04	1.99	1.44E−04	1.99
320	5.66E−05	1.92	7.30E−05	1.99	5.65E−05	2.00	3.62E−05	2.00
640	1.48E−05	1.94	1.83E−05	1.99	1.41E−05	2.00	9.05E−06	2.00
1280	3.83E−06	1.95	4.59E−06	2.00	3.54E−06	2.00	2.27E−06	2.00
2560	9.84E−07	1.96	1.15E−06	2.00	8.85E−07	2.00	5.67E−07	2.00

Example 5.1. Consider the following FODE

$$ {}_C D_{0,t}^{\alpha} y(t) = \frac{\Gamma(6)}{\Gamma(6-\alpha)} t^{5-\alpha} + \frac{36}{\Gamma(5-\alpha)} t^{4-\alpha} - \Gamma(3+\alpha) t^2 - y^2 + (t^5 + \frac{3}{2} t^4 - 2t^{2+\alpha})^2, \tag{5.1} $$

with the initial values

$$ \begin{cases} y(0) = 0, \ y'(0) = 0, & \text{if } 1 < \alpha < 2, \\ y(0) = 0, & \text{if } 0 < \alpha < 1. \end{cases} $$

The exact solution of this equation is $y(t) = t^5 + \frac{3}{2} t^4 - 2t^{2+\alpha}$, and

$$ {}_C D_{0,t}^{\alpha} y(t) = \frac{\Gamma(6)}{\Gamma(6-\alpha)} t^{5-\alpha} + \frac{36}{\Gamma(5-\alpha)} t^{4-\alpha} - \Gamma(3+\alpha) t^2, \ 0 < \alpha < 2. $$

Thus ${}_C D_{0,t}^{\alpha} y(t) \in C^2[0, T]$ for arbitrary $T > 0$ which fulfills the conditions of Theorems 4.1–4.3. For convenience, we check only the rectangle and the trapezoid methods on non-uniform meshes. The predictor–corrector method on non-uniform meshes is omitted here or left to the interested readers. The numerical results are displayed in Tables 1 and 2. According to Theorems 4.1 and 4.3, the orders of convergence of these two methods should be 1 and 2, respectively. From Tables 1 and 2 we can see that the experimental orders of convergence (EOC) support the theoretical convergent orders.

Example 5.2. Consider the following nonlinear equation

$$ {}_C D_{0,t}^{\alpha} y(t) = \begin{cases} \dfrac{2}{\Gamma(3-\alpha)} t^{2-\alpha} - y^2 + (t^2 - t + 1)^2, & \text{for } 1 < \alpha < 2, \\ \dfrac{2}{\Gamma(3-\alpha)} t^{2-\alpha} - \dfrac{1}{\Gamma(2-\alpha)} t^{1-\alpha} & \\ - y^2 + (t^2 - t + 1)^2, & \text{for } 0 < \alpha < 1. \end{cases} \tag{5.2} $$

The initial values has been chosen as follows

$$ \begin{cases} y(0) = 1, \ y'(0) = -1, & \text{if } 1 < \alpha < 2, \\ y(0) = 1, & \text{if } 0 < \alpha < 1. \end{cases} $$

The exact solution of equation (5.2) is $y(t) = t^2 - t + 1$. It is easy to verify that $y(t) \in C^2[0, T]$, but ${}_C D_{0,t}^{\alpha} y(t)$ doesn't belong to $C^2[0, T]$. We can observe from Table 3 that the experimental order of convergence (EOC) on non-uniform meshes is $\min\{2, 1+\alpha\}$ which is still in line with the theoretical analysis in Theorem 4.3 and is even better than the theoretical order when $1 < \alpha < 2$.

The numerical results show that the method on non-uniform meshes has a much higher order of convergence than that on uniform meshes except the case when $\alpha = 0.3$. From this table, the EOCs of non-uniform and uniform meshes are almost the same for $\alpha = 0.3$. But when α increases from 0.3, the regularity of right-hand side of equation (5.2) becomes weaker

Table 3
Absolute errors at $t = 1$ for equation (5.2) using predictor–corrector scheme (2.5).

Meshes	N	$\alpha = 0.30$	EOC	$\alpha = 0.90$	EOC	$\alpha = 1.30$	EOC	$\alpha = 1.90$	EOC
Non-uniform	10	1.75E−01		2.20E−02		5.20E−03		8.43E−03	
	20	7.68E−02	1.19	5.93E−03	1.89	1.10E−03	2.24	2.11E−03	2.00
	40	2.72E−02	1.50	1.57E−03	1.92	2.23E−04	2.31	5.10E−04	2.05
	80	8.90E−03	1.61	4.14E−04	1.92	4.34E−05	2.36	1.22E−04	2.07
	160	2.95E−03	1.59	1.10E−04	1.92	8.14E−06	2.41	2.89E−05	2.07
	320	1.01E−03	1.54	2.91E−05	1.91	1.47E−06	2.47	6.88E−06	2.07
Uniform	10	8.57E−02		1.18E−02		1.50E−04		5.04E−02	
	20	2.86E−02	1.59	4.12E−03	1.52	1.97E−04	−0.39	2.37E−02	1.09
	40	9.07E−03	1.66	1.57E−03	1.39	1.12E−04	0.82	1.11E−02	1.10
	80	2.96E−03	1.61	6.47E−04	1.28	4.50E−05	1.31	5.18E−03	1.10
	160	1.01E−03	1.55	2.79E−04	1.21	1.61E−05	1.48	2.42E−03	1.10
	320	3.58E−04	1.50	1.24E−04	1.17	5.42E−06	1.57	1.13E−03	1.10

Table 4
Absolute errors at $t = 1$ for equation (5.3) using trapezoid scheme (2.3).

Meshes	N	$\alpha = 0.20$	EOC	$\alpha = 0.40$	EOC	$\alpha = 0.60$	EOC	$\alpha = 0.80$	EOC
Non-uniform	10	3.41E−04		3.69E−04		5.16E−04		6.24E−04	
	20	6.94E−05	2.30	9.08E−05	2.02	1.32E−04	1.96	1.62E−04	1.94
	40	1.49E−05	2.22	2.25E−05	2.01	3.36E−05	1.98	4.15E−05	1.97
	80	3.37E−06	2.15	5.59E−06	2.01	8.49E−06	1.99	1.05E−05	1.98
	160	7.85E−07	2.10	1.39E−06	2.00	2.13E−06	1.99	2.64E−06	1.99
	320	1.87E−07	2.07	3.48E−07	2.00	5.35E−07	2.00	6.63E−07	2.00
Uniform	10	1.29E−03		1.27E−03		9.61E−04		6.37E−04	
	20	5.46E−04	1.24	4.67E−04	1.44	3.07E−04	1.65	1.79E−04	1.84
	40	2.34E−04	1.22	1.74E−04	1.43	9.86E−05	1.64	5.03E−05	1.83
	80	1.01E−04	1.21	6.51E−05	1.42	3.19E−05	1.63	1.42E−05	1.82
	160	4.39E−05	1.21	2.45E−05	1.41	1.04E−05	1.62	4.01E−06	1.82
	320	1.91E−05	1.20	9.24E−06	1.41	3.39E−06	1.62	1.14E−06	1.82

due to the factors $t^{2-\alpha}$ and $t^{1-\alpha}$. In this situation, the convergent order of non-uniform meshes is obviously better than that of uniform meshes.

Example 5.3. Consider the following fractional differential equation

$$ {}_CD_{0,t}^{\alpha} y(t) + y(t) = 0, \quad y(0) = 1, \quad 0 < \alpha < 1. \tag{5.3} $$

The analytical solution is

$$ y(t) = E_{\alpha,1}(-t^{\alpha}), $$

where $E_{\alpha,\beta}(z)$ is the Mittag–Leffler function defined by

$$ E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \quad \alpha, \beta > 0. $$

Here we compare the trapezoid method (2.3) on uniform meshes with that on non-uniform meshes. And we choose the same number n of iteration, for example, $n = 10$ for both methods. From Table 4, one can see that the convergent order of this method on non-uniform meshes is still the second order though $y(t)$ is not smooth at $t = 0$, and that the present method on non-uniform meshes has much better performance than that on uniform meshes.

In the following, we give the last example.

Example 5.4. Consider the following time-fractional subdiffusion equation

$$ \begin{cases} {}_CD_{0,t}^{\alpha} u(x,t) = \Delta u + f(x,t), & (x,t) \in (a,b) \times [0,T], \\ u(x,0) = \phi(x), & x \in (a,b), \\ u(a,t) = \psi_1(t), \ u(b,t) = \psi_2(t), & t \in [0,T]. \end{cases} \tag{5.4} $$

where Δ is Laplacian and $\alpha \in (0,1)$.

Table 5
Errors in maximum norm at $t = 1$ for equation (5.4) using trapezoid scheme (5.7).

Meshes	N	$\alpha = 0.25$	EOC	$\alpha = 0.4$	EOC	$\alpha = 0.6$	EOC	$\alpha = 0.75$	EOC
Non-uniform	10	5.96E−05		7.43E−05		6.16E−05		3.28E−05	
	20	1.60E−05	1.89	1.94E−05	1.94	1.57E−05	1.98	8.31E−06	1.98
	40	4.17E−06	1.94	4.92E−06	1.98	3.87E−06	2.02	2.01E−06	2.05
	80	1.02E−06	2.03	1.19E−06	2.05	8.98E−07	2.11	4.28E−07	2.23
	160	1.96E−07	2.38	2.33E−07	2.35	1.55E−07	2.53	3.63E−08	3.56
Uniform	10	6.03E−05		9.65E−05		1.41E−04		1.63E−04	
	20	1.74E−05	1.79	2.93E−05	1.72	4.95E−05	1.51	6.62E−05	1.30
	40	5.01E−06	1.80	9.06E−06	1.69	1.82E−05	1.44	2.85E−05	1.22
	80	1.41E−06	1.83	2.83E−06	1.68	6.97E−06	1.39	1.27E−05	1.16
	160	3.55E−07	1.99	8.69E−07	1.71	2.73E−06	1.35	5.83E−06	1.13

Here we use the centered difference scheme to numerically approximate Δu in (5.4) with the spatial stepsize h, then the following fractional ordinary differential equation can be obtained

$$ {}_{C}D_{0,t}^{\alpha}u(x,t) = \frac{1}{h^2}(u(x+h,t) - 2u(x,t) + u(x-h,t)) + f(x,t). \tag{5.5}$$

Denote $x_i = a + ih,\ \ i = 0, 1, \cdots, L$, and $h = \frac{b-a}{L}$. According to the trapezoid approach (2.3) for equation (1.1), we can derive that for equation (5.5) as follows with the giving initial-boundary condition.

$$\begin{aligned} u_i^{k+1} = \phi(x_i) + \sum_{j=0}^{k+1} \widetilde{w}_{j,k+1} f_i^j + \frac{1}{h^2}\sum_{j=0}^{k+1}\widetilde{w}_{j,k+1}(u_{i+1}^j - 2u_i^j + u_{i-1}^j),& \\ 1 \le i \le L-1, \quad 0 \le k \le N-1,& \end{aligned} \tag{5.6}$$

where u_i^j is the approximation of $u(x_i, t_j)$, $f_i^j = f(x_i, t_j)$ and $\widetilde{w}_{j,k+1}$ is given by equation (2.4).

This implicit scheme can be rewritten in another form as below

$$\begin{aligned} &-\frac{1}{h^2}\widetilde{w}_{k+1,k+1}u_{i+1}^{k+1} + \Big(1 + \frac{2}{h^2}\widetilde{w}_{k+1,k+1}\Big)u_i^{k+1} - \frac{1}{h^2}\widetilde{w}_{k+1,k+1}u_{i-1}^{k+1} \\ &\quad = \phi(x_i) + \sum_{j=0}^{k+1}\widetilde{w}_{j,k+1}f_i^j + \frac{1}{h^2}\sum_{j=0}^{k}\widetilde{w}_{j,k+1}(u_{i+1}^j - 2u_i^j + u_{i-1}^j) \end{aligned} \tag{5.7}$$

where $i = 1, 2, \cdots, L-1$ and $k = 0, 1, \cdots, N-1$.

Now, we give the numerical experiment using Thomas algorithm based on scheme (5.7). If we take $u(x,t) = \sin(\pi x)(t^{0.8} + 1)$ as the solution of equation (5.4), then the source item and definite conditions are read as

$$\begin{cases} f(x,t) = \sin(\pi x)\Big(\pi^2(t^{0.8}+1) + \dfrac{\Gamma(1.8)}{\Gamma(1.8-\alpha)}t^{0.8-\alpha}\Big), & (x,t) \in (0,1)\times[0,T], \\ u(x,0) = \sin(\pi x), & x \in (0,1), \\ u(0,t) = 0, \ \ u(1,t) = 0, & t \in [0,T]. \end{cases}$$

Taking the spatial stepsize as $h = \frac{1}{4096}$, the tests of convergent rate and absolute error on temporal direction are given below. The temporal non-uniform meshes are defined as

$$\tau_k = (k+1)\mu, \ \ 0 \le k \le N-1,$$

where $\mu = \frac{2T}{N(N+1)}$ which are the same as those of Examples 5.1–5.3.

The numerical results are displayed in Table 5. Despite the nonsmoothness of the solution $u(x,t) = \sin(\pi x)(t^{0.8} + 1)$ on temporal direction at the initial time, it is easy to observe that the order of convergence on non-uniform meshes is still the second order which is much higher than that on uniform meshes.

6. Conclusion

In this article, we analyze the stability, convergence and error estimates for three kinds of fractional numerical methods on non-uniform meshes where the non-equidistant stepsize is non-decreasing. The numerical results show that all these methods on non-uniform meshes have better convergence and stability than those on uniform meshes both for FODEs and FPDEs with less smoothness. Particularly, the poorer smoothness of the solution and the right-hand side of equation (1.1), the greater advantages of the non-uniform meshes methods. We can naturally extend the methods and techniques used here to other numerical approaches on non-uniform meshes for fractional (ordinary/partial) differential equations with Caputo derivatives [3,19,21] and/or Riesz derivatives [9–11].

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