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Pareto-optimal equilibrium points in non-cooperative multi-objective optimization problems

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ABSTRACT

In this paper, we consider a class of multi-objective optimization (MOP) problems where the objective holders are independent humans or human-based entities. These problems are indeed game problems, which we call non-cooperative multi-objective optimization problems (NC-MOP). We discuss that for such problems, the Pareto-Optimal (PO) solutions are not necessarily valid as they primarily require Nash equilibrium (NE) solutions. Instead, we suggest that a new solution concept of the Pareto-optimal Equilibrium (POE) point could be adopted. Such a solution is, in particular, important in engineering design and articulation of new rules and protocols among independent entities.

This paper reviews all relevant works that approach the POE concept and investigates the interplay between game problems and multi-objective optimization problems. We present illustrative examples to deepen our understanding of where a POE solution is achievable, as this is not always the case.

1. Introduction

The two disciplines of game theory and multi-objective optimization have so far little in common. Game theory is a mathematical tool for the analysis of decision problems related to cooperative and non-cooperative players. In a non-cooperative framework, which is of interest to us, any player seeks to maximize his payoff. In this case, the solution concept is Nash equilibrium (NE), which is a solution in which no player may improve their expected payoff by changing his strategy profile as long as the other players do the same. However, there might exist better solutions where both players obtain better-expected payoffs than those found at the Nash equilibrium points, called the Pareto-optimal (PO) solutions. Still, the PO solutions belong to the discipline of multi-objective optimization theory where NE has no role to play. However, practitioners in both disciplines have been engaged in solving a lot of common problems in various areas such as in design, engineering, economics, management, and social sciences (Aumann, 1987; Binmore, 2007; Ehrgott, 2006; Finus, 2002; Hwang & Masud, 1979; Ignizio, 1976; Jahn, 2011; Luce & Raiffa, 2012; Roger, 1991; Nash, 1951; Osborne & Rubinstein, 1994; Schelling, 1980; Starr & Zeleny, 1977; Von & Morgenstern, 1944; Zeleny, 1982).

The question is whether a multi-player decision problem is different from a multi-objective decision problem. True is that the conflicts among goals are similar to the conflicts among players, but how is that the solution concepts are essentially different. It has been recognized that a solution to a multi-objective optimization problem seeks a balance between several conflicting objectives, and a solution to a multi-player non-cooperative game problem seeks a balance between conflicting parties. Still, the two solution concepts of NE and PO are different due to the nature of interactions between goal holders or players. Whether the type of interaction is cooperative or non-cooperative, the solution concept is different, i.e., PO and NE, respectively. What is the problem then?

Among the general class of multi-objective optimization problems (MOP), one can see cases where goal holders are humans or human-based entities. However, these problems are, in fact, game problems rather than optimization problems. These problems, more accurately, can be called *non-cooperative MOP* (NC-MOP), as such that players' strategies can be picked from the common feasible zone, i.e., the coupled feasible set. Hence, NC-MOP problems are distinguished from their classical counterpart of MOP problems.

There are a lot of examples of NC-MOPs. Consider, for instance, the Caspian Sea, which is a multi-purpose reservoir with shared resources (e.g., oil resources, hydropower, and fisheries) among five independent

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Table 1
Review of past papers on POE concept in NC-MOP problems.

1. (Vincent, 1983), Mechanical Engineering (Journal type)	
Considering MOP as a game?	Yes
Introducing a new solution concept?	No. But both Pareto-minimum and Nash equilibrium solutions are proposed to be used together, i.e., conceptually similar to our POE concept.
Proposing a new solution method for finding POE?	No
(Notes)	The paper intends to bring to the attention of design engineers that the so-called design optimization problem is indeed a game problem. Though it is more desirable to have a solution that is Pareto-optimal and Nash equilibrium together. To achieve such solutions, it has been argued that additional information or intervention by a third player might be needed still, how to organize a design team to ensure such solutions is an open problem.
2. (Rao, 1987; Rao & Freiheit, 1991; Rao et al., 1997; Lewis & Mistree, 1998), Structural Engineering (Journal type)	
Considering MOP as a game?	Yes
Introducing a new solution concept?	No. The standard compromised solution concept is used for solving MOPs. However, the importance of the NE solution is also recognized.
Proposing a new solution method for finding POE?	No
(Notes)	Considering MOP problems as game problems are defined graphically. However, stress is not put on non-cooperative games. The importance of the POE solution is recognized. For instance, in (Rao et al., 1997) we read that a good solution is the one that is both individually and collectively stable. That is POE!
3. (Lewis & Mistree, 2001), Mechanical Engineering (Journal type)	
Considering MOP as a game?	Yes. It is called a vector game problem.
Introducing a new solution concept?	No
Proposing a new solution method for finding POE?	Using control theory, an iterative-based algorithm is proposed to estimate the rational reaction set (RRS). However, such an RRS does not always intersect to find a NE solution. The convergence of the algorithm is not guaranteed.
4. (Chanron et al., 2005), Systems design (Journal type)	
Considering MOP as a game?	Yes
Introducing a new solution concept?	No
Proposing a new solution method for finding POE?	Same as (Lewis & Mistree, 2001)An approximation algorithm
5. (Gurnani & Lewis, 2008), Mechanical Engineering (Journal type)	
Considering MOP as a game?	No
Introducing a new solution concept?	No
Proposing a new solution method for finding POE?	Same as (Lewis & Mistree, 2001)
(Notes)	The importance of the POE concept is fully appreciated, but has attributed only to the case of full cooperation among players through full sharing of information, i.e., objectives, constraints, gradients, etc.
6. (Facchinei & Kanzow, 2010), Operations Research (Journal type)	
Considering MOP as a game?	Yes, in a sense depicted in Fig. 1
Introducing a new solution concept?	Generalized Nash Equilibrium (GNE)
Proposing a new solution method for finding POE?	An optimality conditions-based algorithm is adopted.
(Notes)	The study of GNE is still in its infancy both on the theoretical ground and on the algorithmic side.
7. (Ciucci et al., 2012), Research Engineering Design (Journal type)	
Considering MOP as a game?	Yes
Introducing a new solution concept?	No
Proposing a new solution method for finding POE?	A Nash solution is sought by improving the (Lewis & Mistree, 2001 & Lewis & Mistree, 1998) method
(Notes)	It is not fully characterized that how the solution will approach the Pareto-optimal front when information is shared among the players.
8. (Lee, 2012), Chemical Engineering (Journal type)	
Considering MOP as a game?	Yes
Introducing a new solution concept?	No
Proposing a new solution method for finding POE?	NE solution is adopted as a compromised solution for a MOP problem.
(Notes)	A trial and error method is devised to find the NE point.
9. (Ghotbi et al., 2014), Mathematical Modelling(Journal type)	
Considering MOP as a game?	Yes
Introducing a new solution concept?	No
Proposing a new solution method for finding POE?	A sensitivity analysis based method is developed for finding Stackelberg NE solutions.
(Notes)	The method helps to approximate the nonlinear RRS equations.
10. (Xiao et al., 2015), Engineering & Management (Journal type)	
Considering MOP as a game?	No, but a game between two players is incorporated into a MOP problem, which makes it interesting to our line of inquiry.
Introducing a new solution concept?	No
Proposing a new solution method for finding POE?	No
(Notes)	Some constraints and objective functions are defined reflecting the players' game within a MOP structure and where a weighted function generates the compromised solution.
11. (Rezaei & Kalantar, 2015), Mechanical Engineering (Journal type)	
Considering MOP as a game?	Yes
Introducing a new solution concept?	The optimized Nash equilibrium point is defined exactly in the same sense as POE in our work to act as a compromised solution in a multi-objective optimization problem.

(*continued on next page*)

Table 1 (*continued*)

Proposing a new solution method for finding POE?	The method is a trial and error method.
(Notes)	The solution obtained is not necessarily a Nash equilibrium solution as mentioned in the conclusion of the paper. It is a Nash inspired solution point.
12. (Konak et al., 2017), Computer Engineering (Journal type)	
Considering MOP as a game?	Yes
Introducing a new solution concept?	No. The problem solved is a bi-objective mixed integer programming problem.
Proposing a new solution method for finding POE?	A genetic algorithm is used to approximate the NE and PO solutions
(Notes)	Although the NE solution is not located on the PO front finding a POE solution is advocated as a desirable solution.
13. (Ji et al., 2018), Computer Engineering (Journal type)	
Considering MOP as a game?	Yes, a game problem is considered while each player faces a multi-objective utility function
Introducing a new solution concept?	The Pareto Equilibrium (PE) solution concept is sought, here which is a concept that has already being investigated in game theory circles.
Proposing a new solution method for finding POE?	Two methods are considered: A duality approach and a KKT based approach
(Notes)	Each player has a PO front where points from those fronts are engaged in a non-cooperative game to achieve an NE solution. This solution is not a POE solution. The paper does not deal with the NC-MOP problem.
14. (Monfared et al., 2020), Operations Research (Journal type)	
Considering MOP as a game?	Yes, the MOP problem converts to a game by considering the z-space as a payoff space of a bi-matrix game.
Introducing a new solution concept?	The notion of "induced games", "most preferred solution", and the POE solution is introduced.
Proposing a new solution method for finding POE?	An algorithm is proposed for finding the POE solution for linear bi-objective optimization problems
(Notes)	The definition of POE is a refinement of the PO front or its neighborhood for finding the most preferred solution. This definition is completely different from our approach in this paper.

member states. The truth is that the objectives are conflicting, but it is important to find whether they need a compromised PO solution within a multi-objective framework or a stable equilibrium solution within a game framework. If there is a Pareto-optimal equilibrium (POE) solution for the member states, this would have eased up the long and troubled negotiations they have had for more than four decades (Yusifzade, 2000; Kucera, 2012; Contessi, 2015; RFE/RL, 2018a; RFE/RL, 2018b).

Other NC-MOPs can also be found in the stakeholder at a corporate level; two farmers who share a water resource from the same aquifer; two neighboring countries who are in conflicts due to shared resources (e.g., water, timber, agriculture, recreational activities, etc.), engineers in a multi-disciplinary design project where the designers have to make decisions in private due to organizational barriers or geographical constraints (See Table 1).

The problems mentioned here are examples of NC-MOPs for which PO solutions are not necessarily valid solutions. The basic reason is that NC-MOP problems should be classified under a non-cooperative game framework where the Nash equilibrium solution is relevant. We consider a new alternative solution concept called POE that is different from the Pareto-optimal solution, PO, equilibrium solution, NE, and Pareto-equilibrium solution, PE.

Notice that different solutions may sometimes coincide, but this does not mean that those solution concepts are not distinct. For instance, in zero-sum games, a Nash equilibrium is identical with Pareto-optimal or the maximin solution (Jagannatha Rao, Badhrinath, Pakala, & Mistree, 1997). Further, bargaining Nash solutions are also identical with Pareto-optimal solutions as players are practically cooperating to reach a compromised solution, e.g., in bargaining games (Young & Zamir, 2014). Using a similar argument is that a POE solution, as we considered here, is a distinctive solution concept that we think is valid for a class of non-cooperating MOP problems.

In this paper, we investigate the interplay between NC-MOPs and MOPs by reviewing the history and by delving into solution concepts of relevance, i.e., Nash equilibrium and Pareto-optimality. We present some illustrative examples to find out where a Pareto-optimal equilibrium solution is achievable.

Our main goal here is to show that NC-MOPs are a distinct class of problems. The analysis of mathematical structure that could entail at least a single POE point is of immense importance, which remains to be studied in future works. The paper is organized into the following sections. Section 2 reviews past works from the new perspective of POE. Section 3 considers some theoretical backgrounds for modeling and solving NC-MOP problems. Section 4 explores POE solution for different linear and non-linear problems. Section 5 concludes and presents ways for further studies.

2. Past works

Facchinei and Kanzow (2010) have reviewed generalized Nash equilibrium problems (GNEP) that contain our NC-MOP problems as a sub-class as shown in Fig. 1(a).

GNEP can have many different forms spread between two different structures shown in Fig. 1. On one hand, Fig. 1(a) illustrates a two-player game where strategies are picked from a common feasible area, i.e., a coupled feasible set. Indeed, this is a MOP problem where the objective holders are independent humans or human-based entities. We call these problems non-cooperative multi-objective optimization problems or NC-MOPs. On the other hand, Fig. 1(b) illustrates a two-player game problem where players' feasible sets are fully decoupled, i.e., each player has its domain of strategies. Problems with partial couplings have been also considered where its structures lie between Fig. 1(a) and Fig. 1(b). For instance, in a leader–follower or Stackelberg game, some constraints might be shared between two players to form the partial coupling of shared information. Still, there might be other constraints that solely belong to each player. Each player can then pick a strategy profile that is feasible in both shared and not shared constraint sets.

GNEP in the literature, which have been called differently, e.g., distributed design problems, continuous static games (Vincent, 1983), distributed decision-making (Lewis & Mistree, 1998), pseudo-game, social equilibrium problem, equilibrium programming, coupled constraint equilibrium problem, abstract economy, Facchinei and Kanzow (2010). It is worth noting that GNEP was first introduced by Debreu in (Debreu, 1952), though it was earlier termed "an abstract economy" defined as "In a game, the payoff to each player depends upon the strategies chosen by all, but the domain from which strategies are to be chosen is given to each player independently of the strategies chosen by other players. An abstract economy, then, maybe characterized as a generalization of a game in which the choice of an action by one agent affects both the payoff and the domain of actions of other agents" (Arrow & Debreu, 1954). Such a definition fits the game structure depicted in Fig. 1(b). Our focus, however, in this paper is in Fig. 1(a) where a multi-objective optimization problem is converted into a game problem, i.e., a

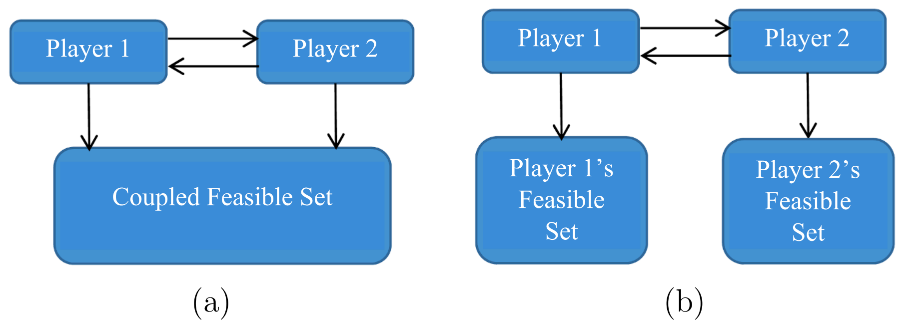

Fig. 1. Two different structure for the generalized Nash equilibrium problems (GNEP): (a) fully coupled games, our NC-MOP problems, and (b) fully decoupled games.

MOP problem is converted into an NC-MOP problem. Handling these problems, however, surfaced since the late nineties by scholars in the engineering field.

Still, our line of inquiry is different from that of the engineering field. Our work looks for POE solutions in NC-MOP problems, which is the class of problems illustrated in Fig. 1(a). In Table 1, we have gathered all the papers of relevance.

We noted here that any works that did not pay attention to the POE concept, whether implicit or explicit have not been reported. Our review starts with the work of Vincent in 1983 (Vincent, 1983). Table 1 is ordered chronologically from 1983 to 2018 and reviews 19 papers We intend to find out whether a MOP has ever be seen or treated as an NC-MOP problem or not. Vincent (1983) here is Nobel in a sense that he has vividly defined an NC-MOP problem and a POE solution concept, named Pareto-minimum Nash Equilibrium. Still, Vincent assumed that a POE solution always exists, which is overlooked as we will show in Section 3. Later researchers have almost neglected the importance of POE solutions, focusing mainly on Nash equilibrium solutions. What we called POE has resurfaced almost two decades later through GNE literature as indicated in row 11 of Table 1.

We think the POE solution is also important to serve as a quality measure to assess organizational efficiency. For instance, consider a design problem where independent experts from different organizations located in different areas are involved in developing a multi-component system. Here, a Nash equilibrium (NE) solution is a valid solution to represent the interactions of independent players; but it might not be an efficient solution. A POE solution, on the other hand, guarantees that such a solution is efficient for all players. A POE solution is both individually and collectively stable. However, finding an efficient POE solution poses new theoretical and methodological challenges as will be considered in the next section.

Now that we have finished reviewing the relevant papers, we may make the following important comments which shall pave the way for future investigations:

1. Most problems dealt with in current literature are of engineering nature.
2. All methods used to solve these problems are essential for a trial and error nature.
3. More theoretically sound methods used, which are KKT-based methods are merely simple heuristic, e.g., the non-linear Jacobi-type method, non-linear Gauss–Seidel-type method, and penalty-type method. These methods suffer from not being globally convergent under reasonable assumptions such as convexity and differentiability. See further details in Facchinei and Kanzow (2010), Dreves, Facchinei, Kanzow, and Sagratella (2011) and Hintermueller and Surowiec (2013).
4. The POE solution concept has been pointed out by only a few papers, i.e., # 1, 11.

Upon such a background of past literature, we now delve into the interplay of a non-cooperative game problem and a multi-objective optimization problem to see where a POE solution is achievable, a high profile quality solution that is both Nash and Pareto.

3. Theoretical backgrounds

To develop our understanding of a POE solution, we start by considering some necessary definitions.

Definition 1. (*Ehrgott, 2006*) A multi-objective optimization problem with p objective functions is formally defined as follows

$$\begin{array}{ll} \max & z(\boldsymbol{x}) = (f_1(\boldsymbol{x}), f_2(\boldsymbol{x}), \cdots, f_p(\boldsymbol{x})) \\ \text{s.t.} & \boldsymbol{x} \in X \end{array} \tag{1}$$

where $X \subseteq \mathbb{R}^n$ is the feasible space or x-space, and, $f_i, i = 1, \cdots, p$, is an n-variable function. The image of X under map z as denoted by Y is called the objective space or z-space, i.e.

$$Y = z(X) = \{\boldsymbol{y} \in \mathbb{R}^p \mid \boldsymbol{y} = \mathrm{z}(\boldsymbol{x})\mathrm{for some}\boldsymbol{x} \in \mathrm{X}\}$$

When X is a polyhedron and objective functions are linear, the problem is called a linear multi-objective optimization problem. In a linear multi-objective optimization problem, if there are two objective functions, i.e. $p = 2$, then the problem is called linear bi-objective optimization (BO) problem.

A feasible point $\widehat{\boldsymbol{x}} \in X$ is called efficient or Pareto-optimal if there is no other $\boldsymbol{x}$ in X such that $f_i(\boldsymbol{x}) \geqslant f_i(\widehat{\boldsymbol{x}})$ for all $i = 1, \cdots, p$.

Definition 2. If the objective holders in problem (1) are human entities or organizations, one can consider each objective holder as a player. Indeed, we have p players. Suppose each player i controls n_i decision variables so that $n = \sum_{i=1}^{p} n_i$. For decision vector $\boldsymbol{x}$ we denote the variables of player i by $\boldsymbol{x}_i$. Let $\boldsymbol{x}^{-i}$ denotes the vector formed by decision variables of all players except those of player i. We use $(\boldsymbol{x}_i, \boldsymbol{x}^{-i})$ as the alternative notation for decision variables vector $\boldsymbol{x} = (x_1, \cdots, x_n)$. Given a fixed decision variable $\widehat{\boldsymbol{x}}^{-i}$ the payoff function $P_i : \mathbb{R}^{n_i} \to \mathbb{R}$ of player i is defined as $P_i(\boldsymbol{x}_i) = f_i(\boldsymbol{x}_i, \widehat{\boldsymbol{x}}^{-i})$. As well as the player i's payoff function, the strategy set of player i depends on the strategies taken by other players. In fact, given a fixed decision variable $\widehat{\boldsymbol{x}}^{-i}$, the strategy set of player i is defined as follows

$$X_i = X_i(\widehat{\boldsymbol{x}}^{-i}) = \{\boldsymbol{x} \in \mathbb{R}^{n_i} : (\boldsymbol{x}, \widehat{\boldsymbol{x}}^{-i}) \in X\}.$$

Recall that X is the feasible space of problem (1). Now, the non-cooperative game corresponding to problem (1) is defined as the triple $G = (X, \{X_i\}_{i=1}^{p}, \{P_i\}_{i=1}^{p})$. This game is a special case of generalized Nash equilibrium problems defined in (Facchinei & Kanzow, 2010).

Definition 3. Consider the game G defined in Definition 2. A point $\overline{\boldsymbol{x}} \in X$ is called a generalized Nash equilibrium of G if for each player $i, \overline{\boldsymbol{x}}_i$

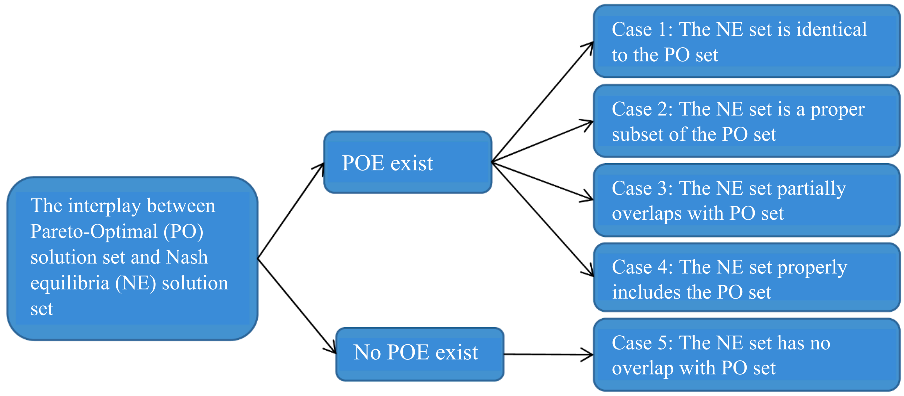

Fig. 2. Exploring the relationship between Pareto-optimal solution set and Nash equilibria set in a NC-MOP in two main cases.

maximizes the payoff function $P_i(\mathbf{x}_i) = f_i(\mathbf{x}_i, \overline{\mathbf{x}}^{-i})$, i.e.

$P_i(\overline{\mathbf{x}}_i) = \max_{\mathbf{x}_i \in X_i(\overline{\mathbf{x}}^{-i})} f_i(\mathbf{x}_i, \overline{\mathbf{x}}^{-i}), \quad i = 1, \cdots, p.$

A generalized Nash equilibrium point $\overline{\mathbf{x}} \in X$ is called a Pareto-optimal equilibrium (POE) solution if it is a Pareto-optimal solution of problem (1).

3.1. POE categories

In Fig. 2, we show the interplay between a PO solution set and a NE solution set for NC-MOP problems to categorize when a POE solution can be exploited, i.e., cases 1, 2, 3, and 4. Altogether we have considered 5 cases of the interplay between the Pareto-optimal solution set and the Nash equilibrium set. This may be seen as an important contribution as future works can be directed toward these categories.

Later in the sequel, we provide examples for each case. It should be noted here that case 4 in Fig. 2 is hypothetically possible, and we could not find an example that lies in this class in our numerical experiments.

Note that solving NC-MOP problems requires novel developments both in theoretical and algorithmic aspects as both are still in their infancy. The solution of the NC-MOP problem can be characterized as the stationary points of a set of ordinary differential equations (ODEs). Where it is difficult to find out under which conditions such a point is asymptotically stable (Facchinei & Kanzow, 2010). One missing criteria in the current literature appear to be that how can one establish a given solution as a valid or even a preferable solution (Facchinei & Kanzow, 2010)? We have contributed to this aim, in this paper, saying that the POE solution concept is going to be a way out.

Note that at present time for finding a POE solution, the NC-MOP problem should be solved twice, i.e., once as a MOP problem to find the PO solution set and once as a game problem to find out the NE solution set.

It is, however, possible to solve an NC-MOP problem directly using methods such as procedures based on Karush-Kahn-Tucker (KKT) optimality conditions as illustrated in Table 1. But, these solutions can not be verified for being POE solutions unless the theory is advanced further to add the necessary and sufficient conditions to the KKT based equations. In the following section, we consider some illustrative examples showing that the existence of POE is dependent upon (1) the mathematical structure of the NC-MOP problem, (2) the choice of variables controlled by the players.

4. Illustrative examples

Our proposal, in this section, is to illustrate how the POE solutions can be found and interpreted considering the four categories considered in Fig. 2. To solve an NC-MOP problem we have to follow three steps:

1. Find the NE solution set by solving the NC-MOP problem as a game,
2. Find the PO solution set by solving the NC-MOP problem as a multi-objective optimization problem,
3. Intersect these two solution sets to find POE solutions if there exist any, i.e. 1, 2, or 3 in Fig. 2.

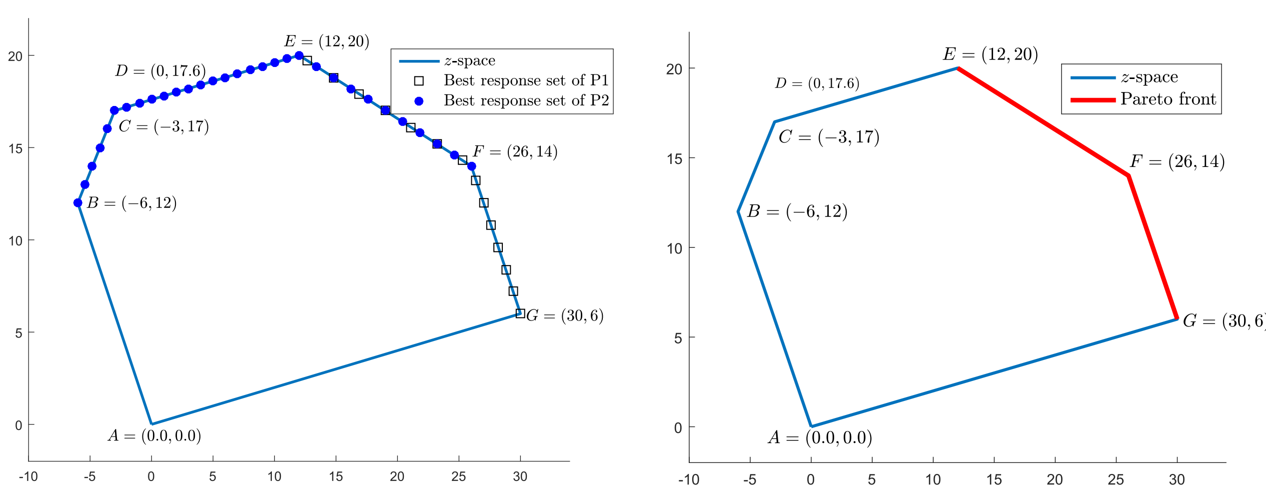

Fig. 3. The best response sets of players for problem (2) (left), and, the Pareto-optimal front of problem (2) (right).

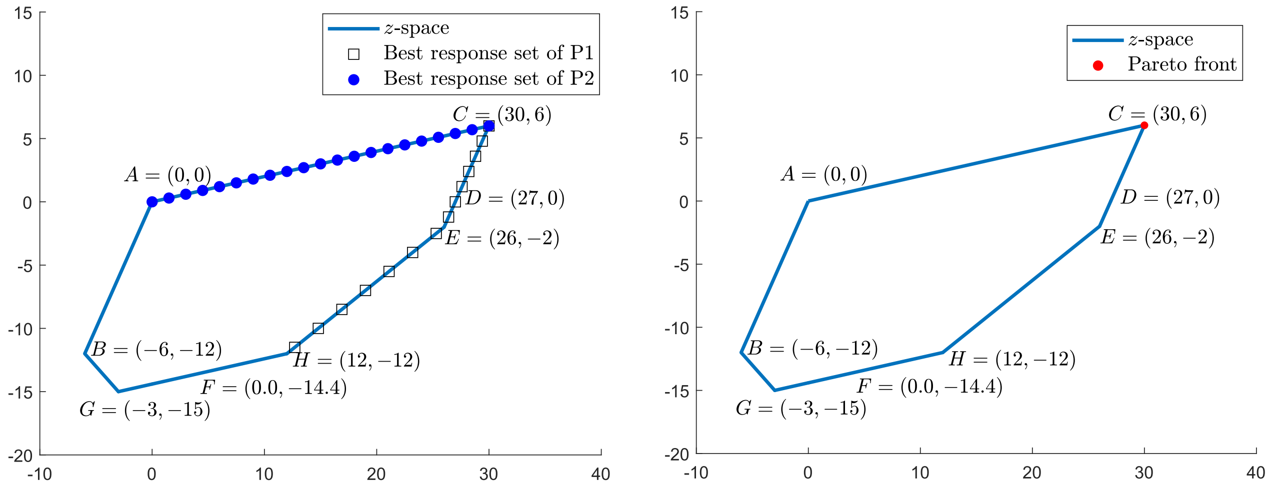

Fig. 4. The best response sets of players for problem (3) (left), and, the Pareto-optimal front of problem (3) (right).

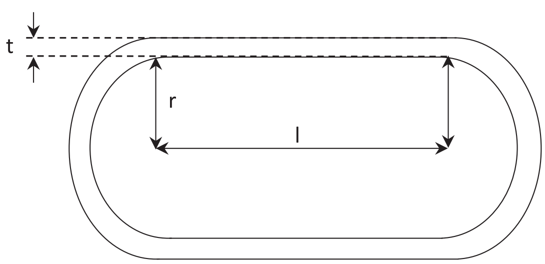

Fig. 5. A thin-walled pressure vessel design problem and design variables.

To illustrate these three steps, we consider some different problems with linear and non-linear objective functions.

4.1. A linear bi-objective problem

Consider the following linear bi-objective optimization problem

$$\begin{aligned} \max \quad & f_1(x_1, x_2) = 5x_1 - 2x_2, \\ \max \quad & f_2(x_1, x_2) = x_1 + 4x_2, \\ \text{s.t.} \quad & (x_1, x_2) \in X, \end{aligned} \tag{2}$$

where $X = \{x \in \mathbb{R}^2 : -x_1 + x_2 \leqslant 3,\ x_1 + x_2 \leqslant 8,\ 0 \leqslant x_1 \leqslant 6, 0 \leqslant x_2 \leqslant 4\}$ and each objective is hold by a player in a non-cooperative game framework so that player 1 controls variable x_1, and player 2 controls variable x_2. Fig. 3 (left) shows the best response sets for both players P1 and P2. In this figure, the best response set for P1 and P2 which have been estimated using the KKT-based method are denoted by unfilled black squares and filled blue circles. As we can see, the intersection of these two best response sets leads to the NE set, which is the line joining vertices E and F in Fig. 3 (left). Also, in Fig. 3 (right), the Pareto-optimal front of problem (2) is shown in red line, which has been solved using the standard method of ϵ-constraint (Ehrgott, 2006). The intersection of the NE set and the Pareto front is the line segment EF. That is, EF is the set of POEs. In this example, the POE set belongs to case 2 in Fig. 2.

Now, let us modify the parameters of the objective functions in problem (2) as shown in the following

$$\begin{aligned} \max \quad & f_1(x_1, x_2) = 5x_1 - 2x_2, \\ \max \quad & f_2(x_1, x_2) = x_1 - 4x_2, \\ \text{s.t.} \quad & (x_1, x_2) \in X \end{aligned} \tag{3}$$

In Fig. 4 (left), the corresponding best response sets for player 1 (P1) and player 2 (P2), are shown. In Fig. 4 (right) the Pareto front is shown.

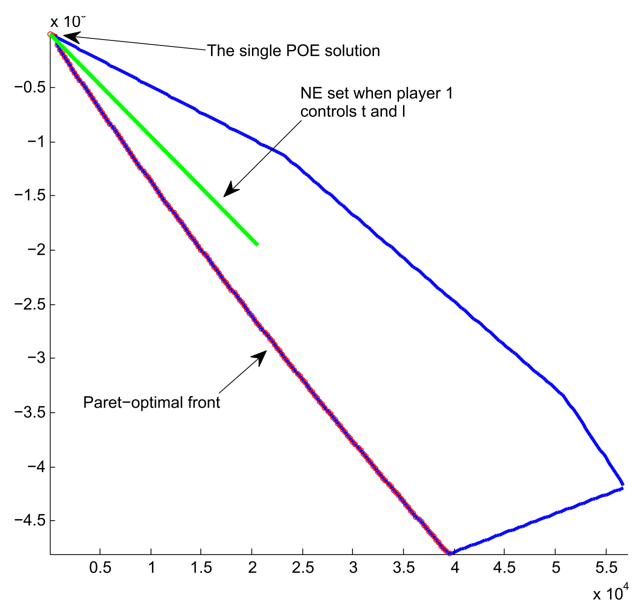

Fig. 6. The z-space and the Pareto-optimal front and the NE set for pressure vessel design problem. It can be seen that Pareto-front and NE set intersect in a single POE point.

Here, we witness that the NE set contains only a single point of $C = (30, 6)$, which is identical to the Pareto front (case 1 in Fig. 2).

4.2. The thin-walled pressure vessel design

Consider the design of a pressure vessel described in Fig. 5 (Xiao, Shao, Gao, & Luo, 2015) with design variables of radius, r, length l and thickness, t. We define a NC-MOP problem with two objective holders such that one objective holder tries to minimize the weight $w(r, l, t) = \rho(4/3\pi(r+t)^3 + \pi(r+t)^2 l - 4/3\pi r^3 - \pi r^2 l)$ by controlling the design variables t and l, and, the second objective holder seeks to minimize the negative of the volume $v(r, l) = 4/3\pi r^3 + \pi r^2 l$ by controlling the design variable r. The pressure vessel should be satisfied with geometry and stress constraints. The problem can now be written as

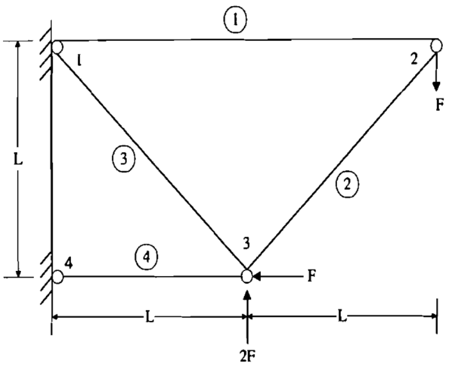

Fig. 7. Four bar truss problem (Ray et al., 2001).

$$
\begin{aligned}
\min \quad & w(r,l,t) = \rho[4/3\pi(r+t)^3 + \pi(r+t)^2 l - 4/3\pi r^3 - \pi r^2 l] \\
\min \quad & v(r,l) = -(4/3\pi r^3 + \pi r^2 l) \\
\text{s.t.} \quad & \frac{pr}{t} \leqslant s_t \\
& 5t - r \leqslant 0 \\
& r + t - 40 \leqslant 0 \\
& l + 2r + 2t - 150 \leqslant 0 \\
& 0.1 \leqslant r \leqslant 36, \quad 0.1 \leqslant l \leqslant 140, \quad 0.5 \leqslant t \leqslant 6
\end{aligned}
$$

where ρ=0.283 lbs/in^3 is the density of the vessel material, $p = 4$klb is the pressure inside the vessel, and, $s_t = 35$klb is the allowable tensile strength of the vessel material. The z-space, the Pareto-optimal, and the NE is shown in Fig. 6. As we can see in this figure, the NE set intersects the Pareto-optimal front in a singleton set, i.e. a POE (case 3 in Fig. 2)

4.3. Four bar truss design

We consider the Four bar truss design problem (Ray, Tai, & Seow, 2001) here in which we have four design variables x_1, x_2, x_3 and x_4. The problem is to minimize structural volume (f_1) and the displacement (f_2) at joint point 2 shown in Fig. 7, subject to the stress constraints:

$$
\begin{aligned}
\min \quad & f_1(\boldsymbol{x}) = L(2x_1 + \sqrt{2}x_2 + \sqrt{2}x_3 + x_4), \\
\min \quad & f_2(\boldsymbol{x}) = \frac{FL}{E}\left(\frac{2}{x_1} + \frac{2\sqrt{2}}{x_2} + \frac{2\sqrt{2}}{x_3} + \frac{2}{x_4}\right), \\
\text{s.t.} \quad & \frac{F}{\sigma} \leqslant x_1, x_4 \leqslant 3\frac{F}{\sigma}, \\
& \sqrt{2}\frac{F}{\sigma} \leqslant x_2, x_3 \leqslant 3\frac{F}{\sigma},
\end{aligned}
\tag{4}
$$

where $F = 10$kN, $E = 2 \times 10^5$kN/cm^2, $L = 200$cm and $\sigma = 10$kN/cm^2. Now, we suppose that player 1 (P1) controls design variables x_1 and x_2, that is $\boldsymbol{x}_1 = (x_1, x_2)$, and player 2 (P2) controls design variables x_3 and x_4, i.e., $\boldsymbol{x}_2 = (x_3, x_4)$. Then, to find the NE solution we need to solve a nonlinear equation system that can be obtained by writing down the optimality conditions for both players P1 and P2. Let λ_i^- and λ_i^+ be KKT multipliers for lower and upper bound constraints on variable x_i, respectively. Then the optimality conditions for both players can be written as follows

Optimality conditions for P1

$$
\begin{aligned}
& 2L - \lambda_1^- + \lambda_1^+ = 0 \\
& \sqrt{2}L - \lambda_2^- + \lambda_2^+ = 0 \\
& \lambda_1^-(\frac{F}{\sigma} - x_1) = 0, \quad \lambda_2^-(\sqrt{2}\frac{F}{\sigma} - x_2) = 0 \\
& \lambda_1^+(3\frac{F}{\sigma} - x_1) = 0, \quad \lambda_2^+(3\frac{F}{\sigma} - x_2) = 0 \\
& \frac{F}{\sigma} \leqslant x_1 \leqslant 3\frac{F}{\sigma}, \quad \sqrt{2}\frac{F}{\sigma} \leqslant x_2 \leqslant 3\frac{F}{\sigma} \\
& \lambda_1^-, \lambda_1^+, \lambda_2^-, \lambda_2^+ \geqslant 0
\end{aligned}
$$

Optimality conditions for P2

$$
\begin{aligned}
& -\frac{2\sqrt{2}FL}{Ex_3^2} - \lambda_3^- + \lambda_3^+ = 0 \\
& -2\frac{FL}{Ex_4^2} - \lambda_4^- + \lambda_4^+ = 0 \\
& \lambda_3^-(\sqrt{2}\frac{F}{\sigma} - x_3) = 0, \quad \lambda_3^+(3\frac{F}{\sigma} - x_3) = 0 \\
& \lambda_4^-(\frac{F}{\sigma} - x_4) = 0, \quad \lambda_4^+(3\frac{F}{\sigma} - x_4) = 0 \\
& \sqrt{2}\frac{F}{\sigma} \leqslant x_3 \leqslant 3\frac{F}{\sigma}, \quad \frac{F}{\sigma} \leqslant x_4 \leqslant 3\frac{F}{\sigma}, \\
& \lambda_3^-, \lambda_3^+, \lambda_4^-, \lambda_4^+ \geqslant 0
\end{aligned}
$$

This system can be solved using different methods. Here, we use the symbolic solver in Mathematica software, and obtain the unique solution of $\boldsymbol{x}_1 = (x_1, x_2) = (1, \sqrt{2})$ and $\boldsymbol{x}_2 = (x_3, x_4) = (3, 3)$. It is easy to see that this solution is not a Pareto-optimal solution for the original problem which is case 5 in Fig. 2 as we have shown in Fig. 8.

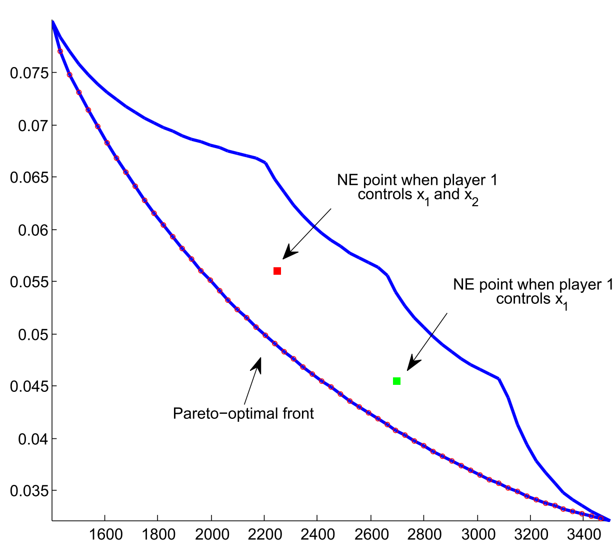

Fig. 8. The z-space and the NE point of four bar truss design problem when the design variables of player 1 are x_1 and x_2 (red), and, when the design variable of player 1 is x_1 (green).

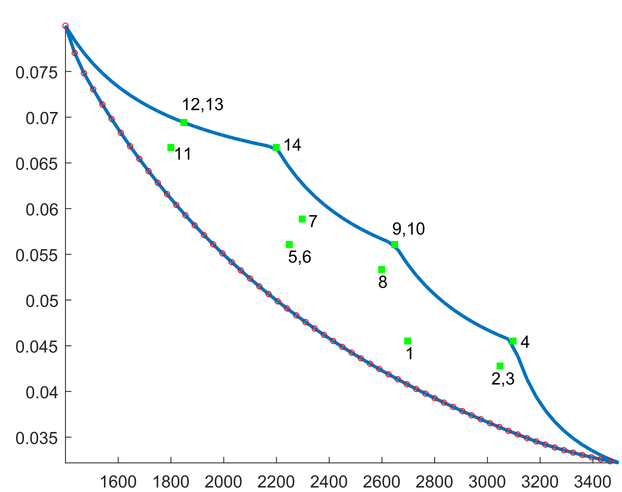

Fig. 9. The z-space of the four bar truss design problem with all cases possible for assigning design variables to the players. The NE point of each assignment is denoted by a green square as well as the corresponding number in Table 2.

Table 2
All possible cases for distributing design variables between two players in the four bar truss design problem.

#	$\mathbf{x}_1$	$\mathbf{x}_2$	NE solution
1	x_1	(x_2, x_3, x_4)	$\mathbf{x}_1 = 1, \mathbf{x}_2 = (3,3,3)$
2	x_2	(x_1, x_3, x_4)	$\mathbf{x}_1 = 3, \mathbf{x}_2 = (\sqrt{2},3,3)$
3	x_3	(x_1, x_2, x_4)	$\mathbf{x}_1 = 3, \mathbf{x}_2 = (3,\sqrt{2},3)$
4	x_4	(x_1, x_2, x_3)	$\mathbf{x}_1 = 3, \mathbf{x}_2 = (3,3,1)$
5	(x_1,x_2)	(x_3, x_4)	$\mathbf{x}_1 = (1,\sqrt{2}), \mathbf{x}_2 = (3,3)$
6	(x_1,x_3)	(x_2, x_4)	$\mathbf{x}_1 = (1,3), \mathbf{x}_2 = (\sqrt{2},3)$
7	(x_1,x_4)	(x_2, x_3)	$\mathbf{x}_1 = (1,3), \mathbf{x}_2 = (3,1)$
8	(x_2,x_3)	(x_1, x_4)	$\mathbf{x}_1 = (3,\sqrt{2}), \mathbf{x}_2 = (\sqrt{2},3)$
9	(x_2,x_4)	(x_1, x_3)	$\mathbf{x}_1 = (3,\sqrt{2}), \mathbf{x}_2 = (3,1)$
10	(x_3,x_4)	(x_1, x_2)	$\mathbf{x}_1 = (3,3), \mathbf{x}_2 = (\sqrt{2},1)$
11	(x_1,x_2,x_3)	x_4	$\mathbf{x}_1 = (1,\sqrt{2},\sqrt{2}), \mathbf{x}_2 = 3$
12	(x_1,x_2,x_4)	x_3	$\mathbf{x}_1 = (1,\sqrt{2},3), \mathbf{x}_2 = 1$
13	(x_1,x_3,x_4)	x_2	$\mathbf{x}_1 = (1,3,\sqrt{2}), \mathbf{x}_2 = 1$
14	(x_2,x_3,x_4)	x_1	$\mathbf{x}_1 = (3,\sqrt{2},\sqrt{2}), \mathbf{x}_2 = 1$

Table 3
All possible cases for distributing design variables between two players in the max version of the four bar truss design problem. Note that the objective functions have been negated.

#	$\mathbf{x}_1$	$\mathbf{x}_2$	NE solution
1	x_1	(x_2, x_3, x_4)	$\mathbf{x}_1 = 3, \mathbf{x}_2 = (3,\sqrt{2},1)$
2	x_2	(x_1, x_3, x_4)	$\mathbf{x}_1 = 1, \mathbf{x}_2 = (3,\sqrt{2},1)$
3	x_3	(x_1, x_2, x_4)	$\mathbf{x}_1 = 1, \mathbf{x}_2 = (\sqrt{2},3,1)$
4	x_4	(x_1, x_2, x_3)	$\mathbf{x}_1 = 1, \mathbf{x}_2 = (\sqrt{2},\sqrt{2},3)$
5	(x_1,x_2)	(x_3, x_4)	$\mathbf{x}_1 = (3,3), \mathbf{x}_2 = (\sqrt{2},1)$
6	(x_1,x_3)	(x_2, x_4)	$\mathbf{x}_1 = (3,\sqrt{2}), \mathbf{x}_2 = (3,1)$
7	(x_1,x_4)	(x_2, x_3)	$\mathbf{x}_1 = (3,\sqrt{2}), \mathbf{x}_2 = (\sqrt{2},3)$
8	(x_2,x_3)	(x_1, x_4)	$\mathbf{x}_1 = (1,3), \mathbf{x}_2 = (3,1)$
9	(x_2,x_4)	(x_1, x_3)	$\mathbf{x}_1 = (1,3), \mathbf{x}_2 = (\sqrt{2},3)$
10	(x_3,x_4)	(x_1, x_2)	$\mathbf{x}_1 = (1,\sqrt{2}), \mathbf{x}_2 = (3,3)$
11	(x_1,x_2,x_3)	x_4	$\mathbf{x}_1 = (3,3,3), \mathbf{x}_2 = 1$
12	(x_1,x_2,x_4)	x_3	$\mathbf{x}_1 = (3,3,\sqrt{2}), \mathbf{x}_2 = 3$
13	(x_1,x_3,x_4)	x_2	$\mathbf{x}_1 = (3,\sqrt{2},3), \mathbf{x}_2 = 3$
14	(x_2,x_3,x_4)	x_1	$\mathbf{x}_1 = (1,3,3), \mathbf{x}_2 = 3$

Now, let us assume that P1 has only x_1 as his design variable under control, i.e., $\mathbf{x}_1 = x_1$, and other variables are controlled by P2, i.e., $\mathbf{x}_2 = (x_2,x_3,x_4)$. In this case, the NE solution is again the unique solution $\mathbf{x}_1 = x_1 = 1$ and $\mathbf{x}_2 = (x_2,x_3,x_4) = (3,3,3)$, which is shown in Fig. 8. This problem has no POE solution, which is case 5 in Fig. 2.

Furthermore, we have computed all possible cases for assigning design variables to the players, which are 14 cases. In all cases, the NE set consists of only a single point that does not belong to the Pareto-optimal front, i.e. there is no POE point. In Fig. 9 we depicted all such NE points. Details are shown in Table 2.

The above version of the four-bar truss design problem has no POE, still, we may organize a unique version of the problem that has a POE. Indeed, we consider the NC-MOP with objective functions $-f_1(\mathbf{x})$ and $-f_2(\mathbf{x})$. Table 3 shows unique NE solutions corresponding to all possible choices of assigning design variables to players. As we can see in Fig. 10 there are six choices of splitting control variables that produce three different POE solutions, while the other choices do not lead to a POE.

5. Conclusion

A subclass of traditional multi-objective optimization problems was

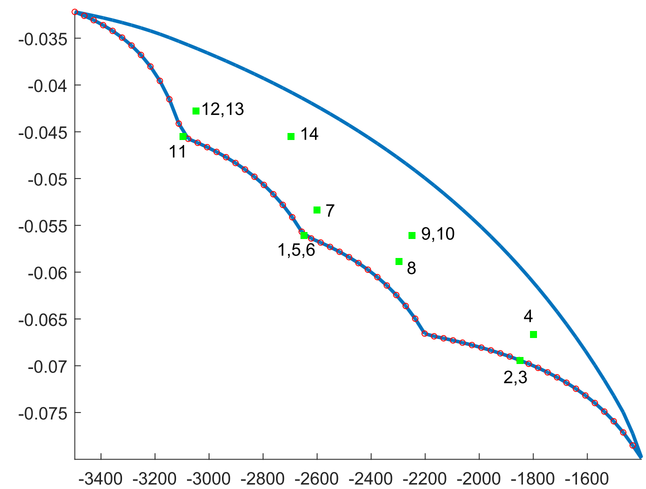

Fig. 10. The z-space of the max version of the four bar truss design problem (i.e. the objective functions have been negated), with all cases possible for assigning design variables to the players. The NE point of each assignment is denoted by a green square as well as the corresponding number in Table 3.

found to illustrate the property of non-cooperative games, i.e. when their objective holders are independent entities. We discussed that for such problems, which we called non-cooperative multi-objective optimization (NC-MOP) problems, a feasible solution must be a Nash equilibrium (NE) and not necessarily a Pareto-optimal (PO) solution. Still, we advocate that an effective solution is a Pareto-optimal equilibrium (POE) solution, and we presented illustrative examples to see how the existence of a POE solution is dependent on the structure and the choice of variables controlled by each player. The main innovation of this paper is our analysis of the interplay between the Nash equilibrium solutions in a non-cooperative game model, and the Pareto-optimal solutions in a multi-objective optimization model. Also, the other contributions are:

- the definition of NC-MOP problems, as a major class of GNEP problem,
- the characterization of POE solutions, as the main solution concept in NC-MOPs, and
- the division of NC-MOP problems into 5 subclasses.

Further researches could be followed to delve deeper into the theory of NC-MOPs, i.e., by finding out how an NC-MOP problem could be structured to have a unique POE solution, i.e., to find out how the problem structure could influence the POE solutions, whether this is unique, many or any. We think when we find that association between structure and solution, we may take advantage of it in organizing collaborative and competitive teams, in particular in engineering designs where the diverse teams should get together from different branches of the same company or different companies. Efficient solution methods are also needed to be developed for every 5 cases of problems, which we have characterized. The managerial implication of this research is appeared to be that practitioners will soon find out that most of the PO solutions they have adopted so far needed to be revised to become POEs.

CRediT authorship contribution statement

Mohammadali Saniee Monfared: Conceptualization, Validation, Methodology, Writing - original draft, Supervision. **Sayyed Ehsan Monabbati:** Conceptualization, Validation, Methodology, Software, Formal analysis. **Atefeh Rajabi Kafshgar:** Conceptualization, Visualization.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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